W P S (DEPR) : 4 / 2011

RBI WORKING PAPER SERIES

Estimation of Portfolio Value at Risk using Copula

Indrajit Roy

DEPARTMENT OF ECONOMIC AND POLICY RESEARCH

APRIL 2011

Estimation of Portfolio Value at Risk using Copula

Indrajit Roy*

Abstract:

In this paper we use high-frequency multivariate data and attempt to model the joint distribution (dependence structure) of daily exchange rate returns of four major foreign currencies (USD, EURO, GBP and Swiss-Franc) against Indian rupees mainly in the copula-GARCH framework. We also compute 1-day, 99% portfolio Value at Risk (VaR) using Monte Carlo simulation technique for seven multivariate models, which were used to model the dependence structure of the four exchange rate returns. We also compare the performances of these multivariate models based on the goodness of in-sample fit as well as backtesting of VaR results. It is observed that multivariate normal distribution does not fit well the joint distribution of four exchange rate returns under consideration, and also number of exceptions raised in backtesting of VaR estimate are exceptionally high and also unconditional coverage test (binomial test/ kupiec test) and conditional coverage test (christoffersen test) suggest that the VaR estimate is inaccurate. In contrast, VaR estimate based on other six multivariate models produce acceptable VaR estimate. However, among all these seven models Clayton copula model and multivariate student's t distribution after transforming individual exchange rate returns to student's t distribution (Hull-White transformation) produce least number of exceptions in back testing of VaR estimate.

JEL Classification: C02; C53; C58; G17

Keywords: Copula, GARCH, Exchange rate, Value at Risk (VaR), Monte Carlo Simulation

---

*Asst. Adviser, Department of Statistics and Information Management, Reserve Bank of India, C-8, Bandra-Kurla Complex, Bandra(E), Mumbai – 400051. email:. The views expressed in this paper are those of the author alone and not of the institution to which he belongs.

Estimation of portfolio Value at Risk using Copula

1. Introduction

Recently, the interaction and dependence among the stock markets, exchange rates and interest rates, both locally and cross border, have become stronger than before, mainly on account of greater integration of financial markets, financial innovations, technology innovation which facilitate in massive flow of information among investors and policy makers. A better understanding of the dependence of asset prices is important for proper risk measurement which also helps in deriving full benefit of portfolio diversification by bank/financial analysts. For example, let us assume that the joint distribution of asset prices is skewed, such that there is higher probability of dependence in the left tail than in the right tail. Then, if we assume a symmetric joint distribution to measure the risk (such as value at risk), the assessment will be incorrect since downside and upside risks are different. Value at Risk (VaR) is widely used as a measure of risk of an asset or of a portfolio of asset. The 100α% 1-day ahead VaR (let us say $\lambda_{\alpha,t}$) is defined as $P[r_t <= \lambda_{\alpha,t} \mid r_{t-1}] = \alpha$; where r_t is the expected return of the portfolio in day 't', r_{t-1} is the observed return of the portfolio in day 't-1'. For example, if there is very little chance, say 5% probability that tomorrow's expected losses for a portfolio would be greater than ₹100, and then ₹100 is the 1-day 5% VaR for the portfolio.

In this paper we use high-frequency multivariate data and attempt to model the joint distribution (dependency structure) of daily returns of four major foreign currencies (INR-USD, INR-EURO, INR-GBP and INR-CHF) against Indian rupees. Like in many previous works (as discussed in Patton (2002), Alexander et al (2005)), the modelling framework we adopt here is mainly a copula-GARCH model. In particular, we use ARMA-GARCH specification (ARMA for mean specification and GARCH for volatility modeling) to filter the deterministic terms in the daily return series and then model the residuals using number of multivariate statistical models viz. (i) Multivariate normal distribution (ii) Multivariate t-distribution (iii) Converting the individual series so that transformed variables follows Normal distribution (Hull-White transformation) and thereafter fitting these variable to a multivariate normal-distribution. (iv)Transforming the individual series so that transformed variables follows student's t distribution (Hull-White transformation) and thereafter fitting these variable to a multivariate t-distribution. (v) Gauss-copula (vi) Student's t-Copula (vii) Clayton-copula. Thereafter, we compute VaR using Monte Carlo simulation technique for the portfolio with four risk factors (INR-USD, INR-EURO, INR-GBP and INR-CHF exchange rates) of equal weights for each of the seven models of dependency structure as mentioned above. We also compare the performances of these models based on the goodness of in-sample fit (log likelihood values of model fit) to the data as well as back testing of VaR results.

2. Statement of Hypothesis

(a) Daily exchange rate returns of the four major foreign currencies against Indian rupees do not follow Gaussian normal distribution. Therefore, using multivariate normal distribution to model the joint distribution of these daily returns is not appropriate and this may lead to inaccurate estimation of VaR of a portfolio of assets which depends on these exchange rates (risk factors).

(b) Instead of multivariate normal distribution, copula approach of modeling the dependence structure of daily exchange rate returns would produce comparatively better estimation of VaR of a portfolio of assets which depends on these exchange rates (risk factors).

3. Structure of the paper

The paper is organized as follows. Section 4 gives a short literature review on the recent applications of copulas in modeling financial series; Section 5 introduces the modeling dependence structure, where we introduce the copula theory, the copula-GARCH framework and the estimation procedure. In particular, we elaborate seven different multivariate model to model the four exchange rate return series, Section 6 introduces the concept of Value at Risk (VaR) and different techniques to the same, Section 7 describe various methods to compare the performance of models, Section 8 reports estimation results for the seven modeling strategies and makes comparison between them in terms of overall goodness of fit of data and back testing of VaR results and Section 9 concludes.

4. Review of literature

Copula is widely used in modeling the joint distributions because it does not require the assumption of joint normality and allow us to decompose n-dimensional joint distribution into its 'n' marginal distributions and a copula function which glue them together. Sklar (1959) introduced the term copula. A good introduction to the copula theory may be found in the books of Joe (1997) and Nelsen (1999). The papers of Bouye *et al.*(2000), Embrechts, Lindskog and McNeil (2003) present general examples of applications of copula in finance. Cherubini and Luciano (2001) estimated the VaR using the Archimedean family copula and the historical empirical distribution to estimate the marginal distributions; Meneguzzo and Vecchiato (2002) used copula for modeling the risk of credit derivatives, Fortin and Kuzmics (2002) used convex linear combinations of copula for estimating the VaR of a portfolio consists of FSTE and DAX stock indices, Embrechts, McNeil and Straumann (2002) and Embrechts, Hoing and Juri (2003) used copula to model extreme value and risk limits. To model the dependence structure between excess returns of "large cap" and "small cap" stock indices, Patton (2004) makes use of a group of frequently used copulas and focuses on the dependence between two stock indices which are more correlated during the market downturn than they are when in the upturn. The deviation from normality could lead to an inadequate VaR estimate and the portfolio could be either riskier than desired or could be needlessly conservative. To measure this asymmetric dependence, the paper uses exceedence correlation, as suggested by Longin & Solnik (2001) and Ang & Chen (2002), and demonstrates that rotated Gumbel copula yields the highest log likelihood (good fit) among all the copula candidates (including both normal and Student's t copulas) and the same is chosen to model the bivariate distribution of two indices. Long Kang (2007) models the joint distribution of excess returns of four major assets (one year and ten year Treasury bonds and S&P 500 and Nasdaq indices) by a multidimensional copula approach. The modeling framework adopted was a copula-GARCH model where GARCH specification was used to model the marginal distribution of individual assets and then to link the margins together use n-dimensional copula (gauss, t, hierarchical and mixed copula). Nelsen (1998), shows that Archimedean family copula can be used to nest one copula into another copula to form a hierarchical structure. A mixed copula (Tasfack (2006)) is formed by summing up a group of weighted copulas where each copula features dependence between one pair of variables and the sum of the weights is equal to unity. Similar work is also done by Goeij and Marquering (2004) where they model the conditional covariance between stock and bond markets returns by a multivariate GARCH approach. They show strong evidence of heteroskedasticity and asymmetries in the covariance between stock and bond market returns. Tasfack (2006) models dependence structure and extreme co-movements of international equity and bond markets by a regime-switching copula-GARCH model. In one regime, he uses an n-dimensional normal copula to link the marginal distributions and in the other he uses a mixed copula of which each copula component features the dependence structure of a particular pair of variables. The paper empirically demonstrates that dependence between international assets of the same type is high in both regimes while the dependence between equity and bond markets is low even within one country.

5. Modelling dependence structure

To model the dependence structure of the four exchange rates returns, we use two step procedures. At first the stochastic volatility effects of the individual series are modelled by generalized autoregressive conditional heteroskedasticity (GARCH) model (Bollerslev, T, 1986). In particular, we fit univariate ARMA-GARCH models. Thereafter, model the joint dependence structure of the innovations (η) of the respective ARMA-GARCH equations. We use seven different models for the joint dependence structure of the four risk factors (exchange rate returns) i.e. (i) multivariate normal distribution; (ii) multivariate t-distribution; (iii) converting the individual series so that transformed variables follows Normal distribution (Hull-White transformation) and thereafter fitting these variable to a multivariate normal-distribution; (iv) transforming the individual series so that transformed variables follow student's t distribution (Hull-White transformation) and thereafter fitting these variable to a multivariate t-distribution; (v) Gauss-copula; (vi) student's t-Copula; and (vii) Clayton-copula.

5.1 Multivariate distribution: Hull-White transformation:

To model the multivariate distribution, an interesting approach is suggested by Hull and White (1998). If returns are not multivariate normal, they suggested that we can still apply the variance-covariance approach if we transform our returns to make them multivariate normal. We then apply the Monte Carlo technique to transformed returns, and derive the VaR estimates.

Assume there are 'm' different instruments in our portfolio. Let e_{it} be the returns on asset 'i' in period 't', and let G_i be an assumed distribution function for e_{it}. This function will, in general, be time dependent reflecting factors such as GARCH volatility and correlation process. We now transform e_{it} into a new variable (f_{it}) using the transformation:

$f_{it}=N^{-1}[G_i(e_{it})]$ …………………………………………………………………………… (1)

Where N is the standard normal distribution function (or we also could have used student's t distribution). The term in square brackets, $G_i(e_{it})$, is the zth percentile of the assumed distribution function G_i, and f_{it} is the same percentile of the standard normal distribution. Hence, equation (1) transforms the return e_{it} into its standard normal equivalent, f_{it}. We can also invert equation (1) to map the f_{it} back to the original returns, e_{it}, using the reverse mapping:

$e_{it}=G^{-1}_i[N(f_{ij})]$ ....(2)

Equation (1) thus allows us to transform our returns and equation (2) allows us to un-transform them. The function G_i can take any form we like: we can take it to be some particular heavy-tailed distribution, for instance, or we can take it to be empirical distribution function drawn from our original data. Next, we assume that our transformed returns – the f_{it} – are distributed as multivariate normal and we estimate their mean vector and covariance matrix. Hull and White suggest that we use a Monte Carlo method to simulate values of the transformed returns fit based on the estimated mean and variance-covariance parameters. We then use equation (2) to map our simulated f_{it} values to their untransformed counterparts, the e_{it}, to give a set of simulated non-normal returns, and we can estimate the desired risk measures using a standard method (e.g. a non-parametric method based on the simulated series). This approach is easy to implement and can be applied to a much wider set of non-multivariate-normal return distribution. In fact, this is nothing but the gauss-copula (discussed later).

5.2 Copula

Copulas have become a popular tool in multivariate modelling. A copula is a method for associating random variables together, irrespective of their marginal distributions. Copula is a multivariate distribution whose marginal distributions are all uniform distribution over (0, 1). The main purpose of copula is to separate marginal distribution from correlation and this is done by transforming each variable so that it becomes uniformly distributed. For continuous distributions, there is a widely used technique to do so. Let F denote the cumulative distribution function of the random variable X, i.e. $F(x) = P(X<x)$. Then the variable $U=F(X)$ is uniformly distributed [Let u be a number between 0 and 1; then $P(U<u) = P(F(X)<u) = P(X< F^{-1}(u)) = F(F^{-1}(u)) = u$]. Therefore, if we are able to glue together uniform distributions, then that can be termed as copula.

Let $\boldsymbol{X}=(X_1, \ldots, X_n)$ be the random vector with marginal cumulative distribution functions (C.D.F.) $F_1, \ldots, F_n$. The m-dimensional multivariate C.D.F., $\boldsymbol{F}(x_1,\ldots,x_n) = \boldsymbol{P}[X_1 \le x_1,\ldots,X_n \le x_n]$, completely determines the dependence structure of random variables $X_1, \ldots, X_n$. However, its analytic representation is often too complex, making practically impossible its estimation and consequently its use in simulation models. Sklar (1959) first showed that there exists a m -dimensional copula C such that $F(x_1,\ldots,x_m)=C(F_1(x_1),\ldots,F_m(x_m))$.

The use of copula function allows us to overcome the issue of estimating the multivariate C.D.F. by splitting it into two parts:

(a) Determine the margins $F_1, \ldots, F_n$, representing the distribution of each factors; estimate their parameters by fitting with the available data.
(b) Determine the dependence structure of the random variables $X_1, \ldots, X_n$, by means of a suitable copula function.

Copulas provide greater flexibility in modelling the multivariate distribution by allowing us to fit the appropriate marginal to different random variables and then specifying the appropriate copula function that bind these marginal distributions together. In contrast, traditional representations of multivariate distributions require that all random variables have the same marginal distribution. Since a copula can capture dependence structures regardless of the form of the margins, a copula approach to modelling related variables is potentially very useful in risk management. These advantages imply that copulas provide a superior approach to the modelling of multivariate statistical problems. Example and definition of some of the widely used copula such as Gauss copula, student's t-copula, Gumbel copula, Clayton copula are given in Annex I.

6. Value at Risk (VaR)

In order to compute portfolio VaR, we need to identify basic market rates and prices (risk factors) that affect the value of the portfolio. It is necessary to identify a limited number of basic risk factors; otherwise, the complexity of deriving a portfolio level VaR would be difficult. There are three broad methods to compute VaR i.e. Historical Simulation, Variance-Covariance (Parametric) and Monte Carlo technique.

6.1 Historical Simulation

Historical simulation (HS) is simple to implement and requires relatively few assumptions about the statistical distribution of the underlying market factors. HS involves using historical changes in market rates and prices to construct a distribution of potential future portfolio profits and losses and then calculating, for example, the 99%VaR as the loss that is exceeded only 1% of the time. The distribution of profit and losses is constructed by taking the current portfolio and subjecting it to the actual changes in the market factors experienced during each of the last N days (e.g. 250 days). That is N sets of hypothetical values of market factors are constructed using their current values and the changes experienced during the last N periods. Using these hypothetical values of market factors, N hypothetical mark-to-market portfolio values are computed (hypothetical because the current portfolio was not held on each of the last N days). Making use of the actual historical changes in risk factors to compute the hypothetical profits and losses is the important characteristic of historical simulation.

6.2 Variance-Covariance

In Variance-Covariance method of estimation of VaR, it assumes the returns (X) are normally distributed. It requires that we estimate first two moments i.e. mean (μ) and standard deviation (σ) which completely describe the normal distribution. Therefore, 99% VaR is ($\mu - Z_\alpha * \sigma$) i.e. μ-2.33*σ (where $\Pr((X- \mu)/ \sigma < Z_\alpha)=.01$).

6.3 Monte Carlo Simulation

In the case of Monte Carlo technique samples are drawn repeatedly from the random processes governing the prices or returns of the financial instruments we are interested in. For example, if we were interested in estimating a VaR, each simulation would give us a possible value for our portfolio at the end of our holding period. If we take enough of these simulations, the simulated distribution of portfolio values will converge to the portfolio's unknown 'true' distribution, and we can use the simulated distribution of end-period portfolio values to infer the VaR. The simulation process involves a number of specific steps. The first step is to select a model for the stochastic variable(s) of interest. Having chosen our model, we estimate its parameters – volatilities, correlations etc. We then construct the simulated paths for the stochastic variables. Each set of 'random' numbers then produces a set of hypothetical terminals price(s) for the instrument(s) in our portfolio. We then repeat these simulations sufficient times to be confident that the simulated distribution of portfolio values to be a reliable proxy for it. Once that is done, we can infer the VaR from this proxy distribution by using quantile / cumulative distribution function/ percentile.

In contrast to Historical Simulation, the Parametric VaR model imposes a strong theoretical assumption on the underlying properties of data; frequently Normal Distribution is assumed because it is easily understood and can be defined using only the first two moments. Other probability distributions may be used, but at a higher computational cost. However, empirical evidence indicates that asset price returns, in particular the daily price changes, most of the time does not follow Normal Distribution. In the presence of excess kurtosis, failure rate increases when the VaR is estimated by the Gaussian distribution. As a result, multivariate normal distribution assumption of portfolio is frequently unsatisfactory because large changes occurred more frequently than what is predicted under the normality assumption and which lead to underestimation of the portfolio VaR.

Let us assume that the copula C which is a good proxy for the actual multivariate probability distribution of the risk factors of a portfolio has been selected and we are interested in the Value-at-Risk of the portfolio. If we have four risk factors (i.e. four exchange rates) with marginal cumulative distribution function (F_1, F_2, F_3, F_4), we need to generate a set of random variables (X_1,X_2,X_3,X_4) from the selected copula $F(C)$, then ($F_1^{-1}(x_1)$, $F_2^{-1}(x_2)$, $F_3^{-1}(x_3)$, $F_4^{-1}(x_4)$) form one scenario of possible changes of the risk factor. The Monte Carlo method generates N such scenarios, and evaluates the change of value of a portfolio under each of these scenarios. One period VaR with confidence α is computed as the sample α−quantile of the N such scenarios.

7. Comparison of models

To compare the goodness of in-sample fit and performances of the models, we use both the log likelihood values and back testing of results (in terms of number occasion when actual exceed the VaR number). We compute the VaR of the hypothetical portfolio for 200 days using Monte Carlo technique for these seven models and observe the number of occasions of exception (i.e. actual is exceeding the VaR) and some other statistical test as discussed in 7.1.

7.1 Back testing of VaR

7.1.1 Current Regulatory Framework for back testing

According to amendment to 1988 Basle Capital Accord, the capital standards cover all assets in a bank's trading account (i.e., assets carried at their current market value) as well as all foreign exchange and commodity positions wherever located. According to internal models approach (IMA) the capital charges are based on the banks own risk measurement models using the standardizing regulatory parameters of a ten-day holding period (k = 10) and 99% VaR. In other words, market risk capital charge of a bank is based on its own estimate of the potential loss that would not exceed with 99% confidence level over the subsequent two week period. Specifically, a bank's market risk capital charge for time t+1, $MRC_{m,t+1}$ shall be set at the higher of the previous day's VAR, or the average over the 60 business days that is, $MRC_{m,t+1}$=

max [$k\frac{1}{60}\sum_{i=1}^{60} VaRm,t-i$; $VaR_{m,t}(10,1)$] + $SRC_{m,t}$ …………….(3)

where K and $SRC_{m,t}$ are a regulatory multiplication factor and an additional capital charge for the portfolio's idiosyncratic credit risk, respectively. Under the current regulatory frame work, K>=3. The regulatory multiplication factor (k) depends on the number of exceptions (defined as the occasions when $\varepsilon_{t+1} < VaR_{m,t}$ (1,1)) observed over the last 250 trading days. To address the low power of the implied, binomial hypothesis test, the number of such exceptions is divided into three zones. Within the green zone (four or fewer exceptions), a VaR model is deemed "acceptably accurate", and 'k' remains at three, the level specified by the Basle Committee. Within the yellow zone (five through nine exceptions), 'k' increases incrementally with the number of exceptions. Within the red zone (ten or more exceptions), the VaR model is deemed to be inaccurate, and 'k' increases to four. The institution must also explicitly improve its risk management system.

7.1.2 Alternative Evaluation Methods

7.1.2.1 Evaluation of VaR estimates based on the binomial distribution

As discussed in the previous section (7.1.1) that under the current regulatory framework, banks will report their one-day VaR estimates to the regulators, who also verify whether actual portfolio losses exceed these estimates. If we assume that the VaR estimates of the bank are accurate, such observations can be modelled as draws from an independent binomial random variable with a probability of occurrence equal to the specified α % (99 %). As discussed by Kupiec (1995), a variety of tests are available to examine whether the observed probability of occurrence, also known as unconditional coverage, equals α, and the method that regulators have chosen is based on the number of occasions where $\varepsilon_{t+1} < VaR_{m,t}$ (1,1) in a sample. The probability of observing x such exceptions in a sample of size T is

$Pr(X;\alpha,T) = \binom{T}{x} \alpha^{x} (1-\alpha)^{T-x}$……………………………………………………………(4)

Accurate VaR estimates should exhibit the property that their unconditional coverage, measured by $\alpha^* = x/T$, equals the desired coverage level α. **Thus, the relevant null hypothesis is $\alpha^* = \alpha$,** and the appropriate likelihood ratio statistic is

$LR_{uc}(\alpha) = 2\,[\,\log(\alpha^{*X}(1-\alpha^{*})^{T-X}) - \log(\alpha^{X}(1-\alpha)^{T-X})\,]$ …………………………………..(5)

Under the null hypothesis $LR_{uc}(\alpha)$ has an asymptotic $\chi^2(1)$ distribution. That is we reject the null hypothesis at 5% level of significance if the test statistics is greater than 3.841459.

7.1.2.2 Conditional Coverage (Christoffersen 1997)

VaR estimate can be considered as an interval estimate, that is, if $f_{t+k,}$ is the k-step-ahead (forecasts) return probability distribution then α percentile lower left-hand interval is the (100- α)% VaR. Forecast performance can be examined over the sample period with or without reference to the information available at each point in time. The $LR_{uc}(\alpha)$ test (as discussed in 7.1.2.1) is an unconditional test of interval forecasts. However, in the presence of the stochastic volatility, testing the conditional accuracy of interval forecasts becomes important. Moreover, $LR_{uc}(\alpha)$ test fails when the exceptions are clustered in a time dependent fashion. The $LR_{cc}(\alpha)$ test proposed by Christoffersen (1997) is a test of correct conditional coverage. For a given coverage level α, one-step-ahead interval forecasts are formed using model m and are denoted as $V_{mt}(\alpha) = (-\infty, VaR_{mt}(\alpha)]_{t=1}^{T}$. Based on these forecasts ($V_{mt}(\alpha)$) and the observed portfolio returns, the indicator variable $I_{mt}(\alpha)$ generated as given below

$$I_{mt}(\alpha) = \begin{cases} 1 \text{ if } \varepsilon_{t+1} \in V_{mt}(\alpha) \\ 0 \text{ if } \varepsilon_{t+1} \notin V_{mt}(\alpha) \end{cases} \quad \ldots\ldots(6)$$

The $LR_{cc}(\alpha)$ test for correct conditional coverage is formed by combining tests of correct unconditional coverage and independence, and the relevant test statistic is $LR_{cc}(\alpha) = LR_{uc}(\alpha) + LR_{ind}(\alpha)$ which is distributed $\chi^2(2)$. Note that the $LR_{ind}(\alpha)$ statistic is a likelihood ratio statistic of the null hypothesis of serial independence against the alternative of first-order Markov dependence. Under this alternative hypothesis, the likelihood function is $L_A = (1-\pi_{01})^{T_{00}}\pi_{01}^{T_{01}}(1-\pi_{11})^{T_{10}}\pi_{11}^{T_{11}}$ where the T_{ij} is the number of observations in state j after having been in state i the period earlier

$$\pi_{01} = {T_{01}}/{(T_{00}+T_{01})} \quad and \quad \pi_{11} = {T_{11}}/{(T_{10}+T_{11})}$$

Under the null hypothesis of independence, $\pi_{01} = \pi_{11} = \pi$ and the relevant likelihood function is $L0 = (1-\pi)^{T_{00}+T_{10}}(\pi)^{T_{01}+T_{11}}$; where $\pi = (T_{01}+T_{11})/T$. The test statistic is formed as $LR_{ind}(\alpha) = 2[\log L_A - \log L_0]$ which follows $\chi^2(1)$ *asymptotically*.

8. Empirical Analysis

In this study, we use daily data on four exchange rates (INR-USD, INR-EURO, INR-GBP and INR-CHF) series, downloaded from the official source (www.rbi.org.in; www.federalreserve.gov). The sample period is January 2000 to November 2010. The summary statistics is given in Table 1, kernel densities are shown in Fig 1 and daily returns are shown in Figure 2. Various normality tests (Nortest package in 'R' V2.12.0) such as Anderson-Darling normality test, Cramer-von Mises normality test, Lilliefors (Kolmogorov-Smirnov) normality test, Shapiro-Francia normality test, Pearson chi-square normality test suggests that the variables are not normally distributed (Annex II).

Table 1: summary statistics of daily returns of exchange rates

	INR-CHF	INR-EUR	INR-GBP	INR-USD
Mean	0.0002	0.0001	0.0000	0.0000
Median	0.0000	0.0001	0.0001	0.0000
Maximum	0.0433	0.0378	0.0472	0.0394
Minimum	-0.0415	-0.0398	-0.0417	-0.0371
Std. Dev.	0.0076	0.0069	0.0067	0.0040
Skewness	0.0527	0.0078	-0.1582	0.1733
Kurtosis	5.0574	5.1841	7.2376	16.9476
Jarque-Bera	484	544	2061	22215
Probability	0.0000	0.0000	0.0000	0.0000
Observations	2739	2739	2739	2739

Figure 1: Kernel densities of daily returns of exchange rates

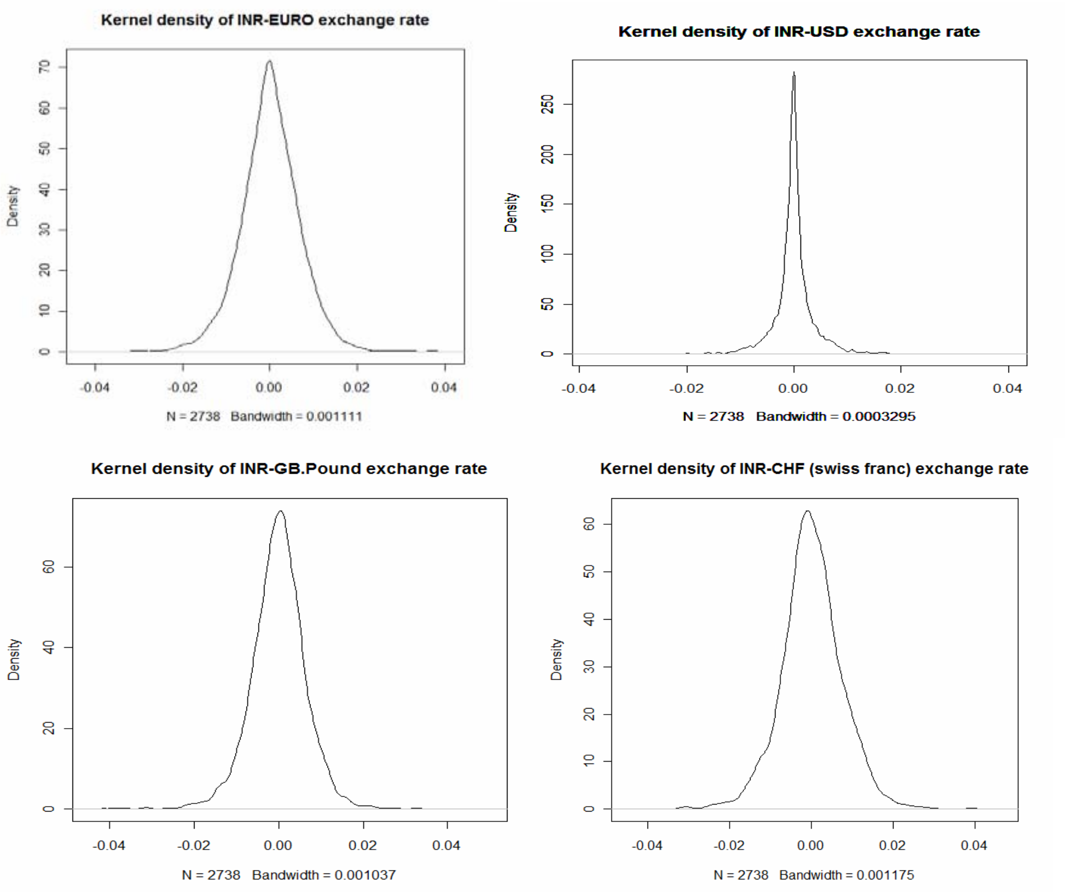
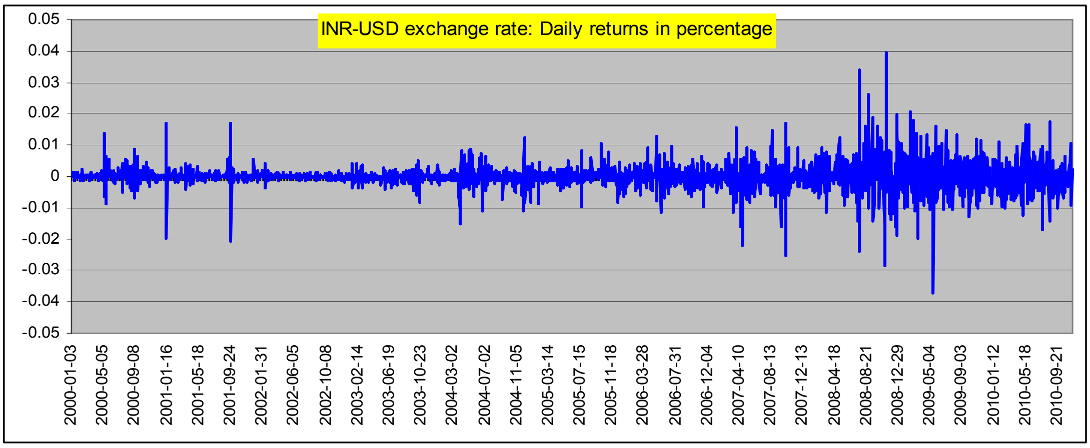
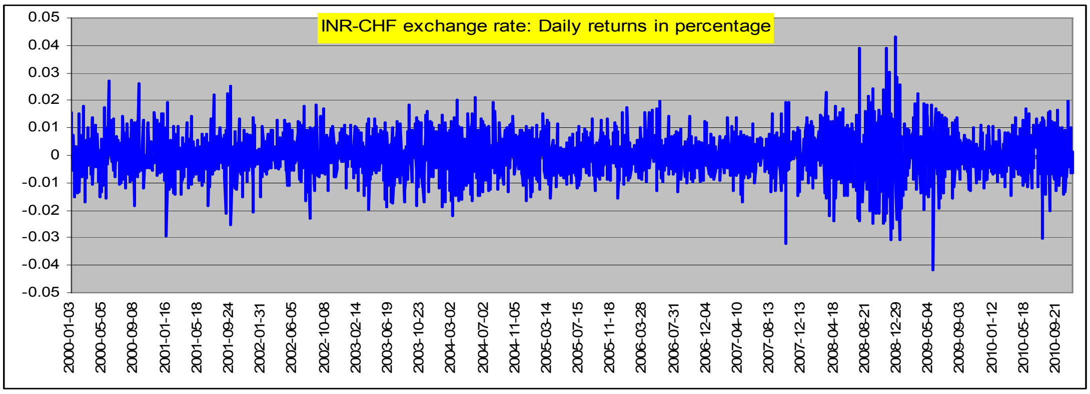
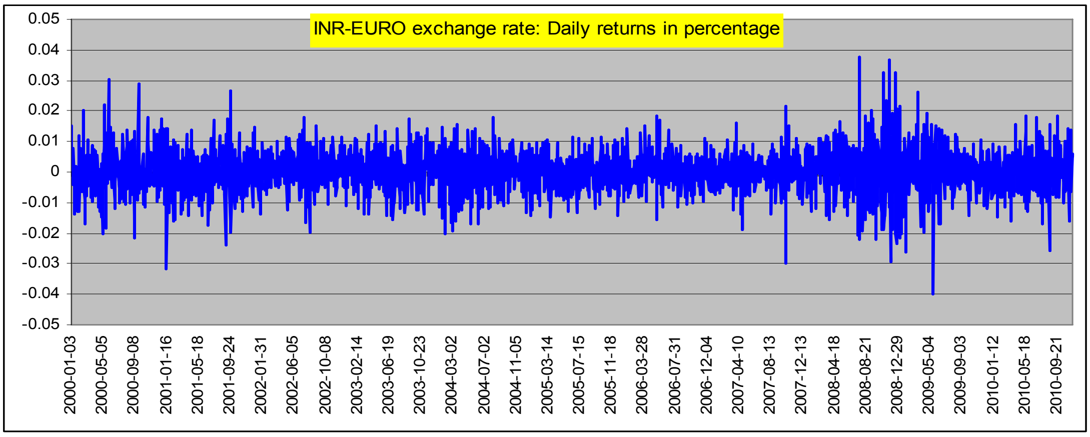
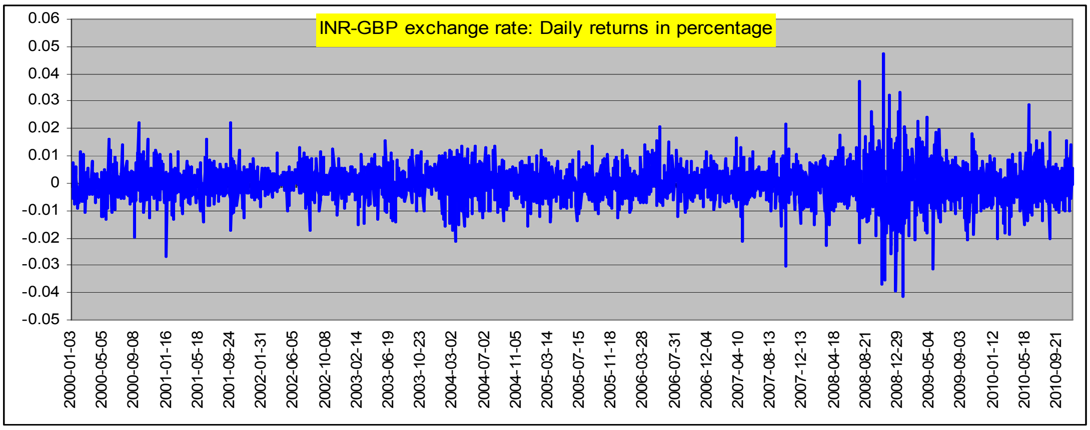


Figure 2: Plotted daily returns of the four exchange rates









8.1 Modelling marginal distributions

Results of the ARMA-GARCH models for the daily returns of INR-USD, INR-EURO, INR-GBP and INR-CHF exchange rates are given in Annex III. The ARMA_GARCH model for the conditional mean and conditional variance of these four series are as under:

INR-USD
USD = -3.10357879533e-05 + [MA(1)=-0.0946539736752] ………………………….(7)
GARCH = 2.19530647185e-06 + 0.862185107635*GARCH(-1)

INR-EURO
EUR = 0.000120101838983 + [AR(1)=-0.0545592135554] ………………………….(8)
GARCH = 7.20273356275e-07 + 0.0517524042758*RESID(-1)^2 + 0.933656466512*GARCH(-1)

INR-SWISS Franc
CHF = 0.000115982272748 + [MA(1)=-0.07135] ……………………………………..(9)
GARCH = 7.55810434168e-07 + 0.0451261892429*RESID(-1)^2 + 0.94120692432*GARCH(-1)

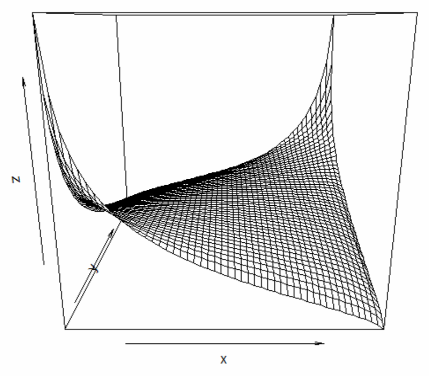
INR-GBP

GBP = 7.64605751336e-05 + [MA(1)=-0.0617194170291] ……………………….(10)
GARCH = 5.9605825624e-07 + 0.052704390331*RESID(-1)^2 + 0.933170303731*GARCH(-1)

8.2 Modelling dependence structure

To model the dependence structure of the four exchange rates returns, we construct the joint dependence structure of the innovations (η) of the respective GARCH equation. We use seven different models for the combined dependence structure of the four risk factors (exchange rate returns) i.e. (i) multivariate normal distribution; (ii) multivariate t-distribution; (iii) converting the individual series so that transformed variables follows Normal distribution (Hull-White transformation) and thereafter fitting these variable to a multivariate normal-distribution; (iv) transforming the individual series so that transformed variables follows student's t distribution (Hull-White transformation) and thereafter fitting these variable to a multivariate t-distribution; (v) Gauss-copula; (vi) t-Copula; and (vii) Clayton-copula. We have estimated the model using the QRMlib package in **'R' version 2.12.0** (http://cran.r-project.org/). The fitted models are shown in Figure 3.a to 3.x. The 'R' script used to calibrate and back test the model is given in Annex IV. Once the model parameters are estimated, we have used the Monte Carlo simulation technique to calculate the VaR of the portfolio. That is, we draw a large number of random observations (2500) for each of the four exchange rates from each of the seven calibrated multivariate models (i) to (vii) as described above and calculate the average (assuming equal weight of the risk factors in the portfolio). If we sort these averages (2500) for each of the models separately in descending order 99% VaR would be the 99 percentile of these sorted series. To assess the performance of VaR of these seven models we performed a backtesting for last 200 days (10-Dec-2009 to 27-Sep-2010).

Figure 3.a Gauss Copula: INR-USD and INR-EURO   Figure 3.b t-Copula: INR-USD and INR-EURO



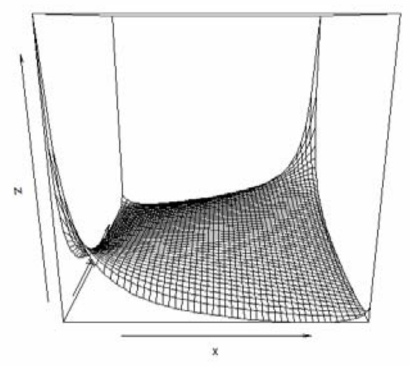
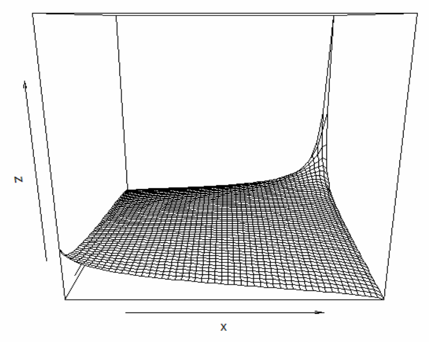


Figure 3.c Gumble-Copula: INR-USD and INR-EURO   Figure 3.d Clayton -Copula: INR-USD and INR-EURO



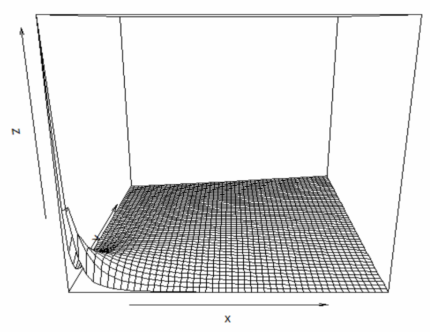
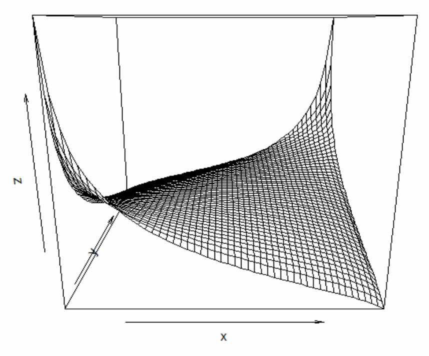


Figure 3.e Gauss Copula: INR-USD and INR-CHF   Figure 3.f t-Copula: INR-USD and INR-CHF



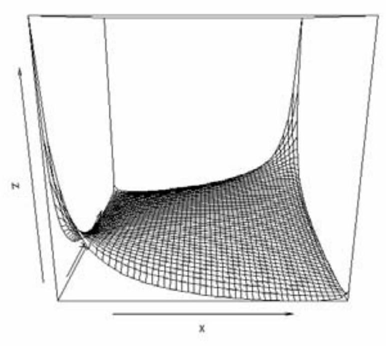
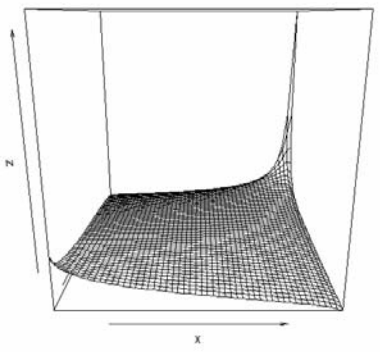


Figure 3.g Gumble -Copula: INR-USD and INR-CHF   Figure 3.h Clayton -Copula: INR-USD and INR-CHF



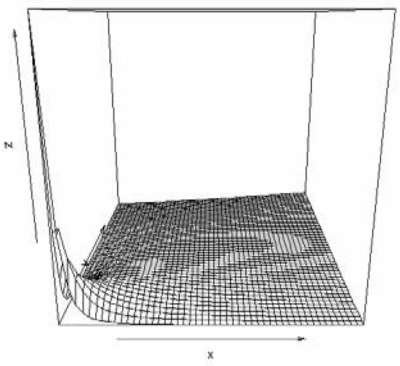
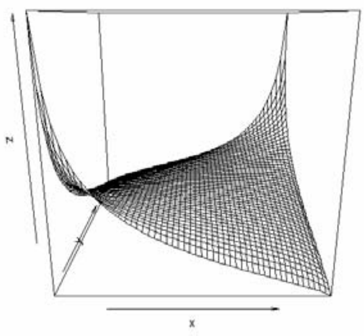


Figure 3.i Gauss Copula: INR-USD and INR-GBP   Figure 3.j t-Copula: INR-USD and INR-GBP



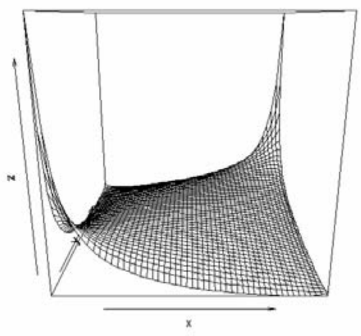
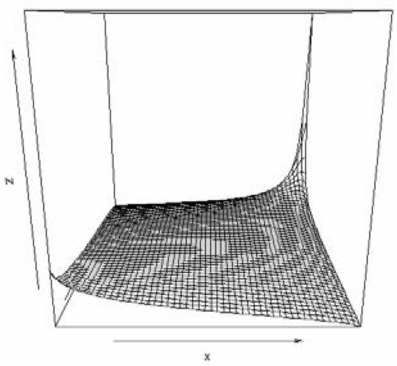


Figure 3.k Gumble -Copula: INR-USD and INR-GBP Figure 3.l Clayton -Copula: INR-USD and INR-GBP



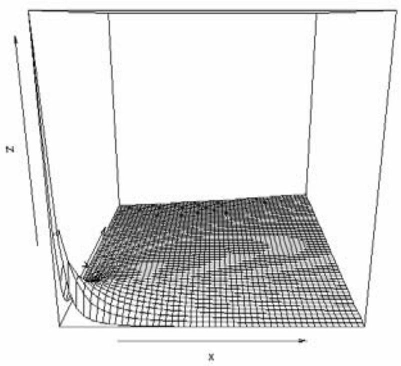
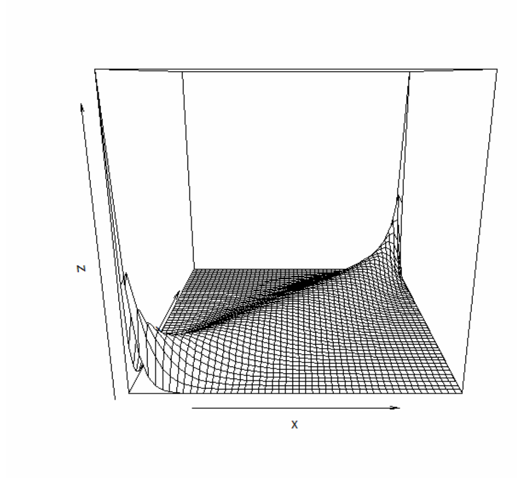


Figure 3.m Gauss Copula: INR-EURO and INR-CHF   Figure 3.n t-Copula: INR-EURO and INR-CHF



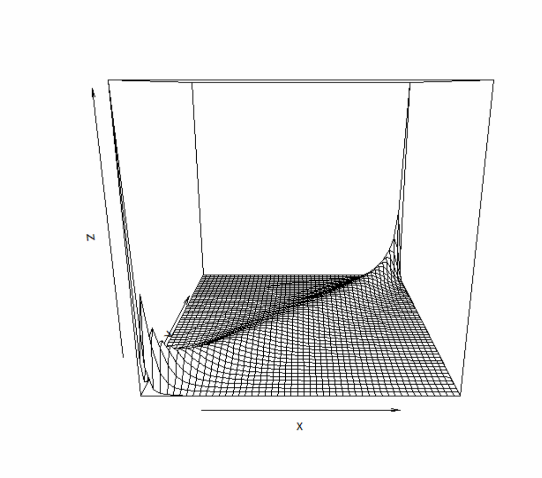
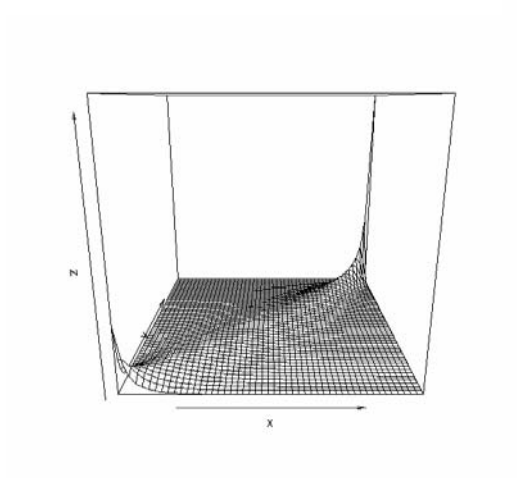


Figure 3.o Gumble -Copula: INR-EURO and INR-CHF Figure 3.p Clayton -Copula: INR-EURO and INR-CHF



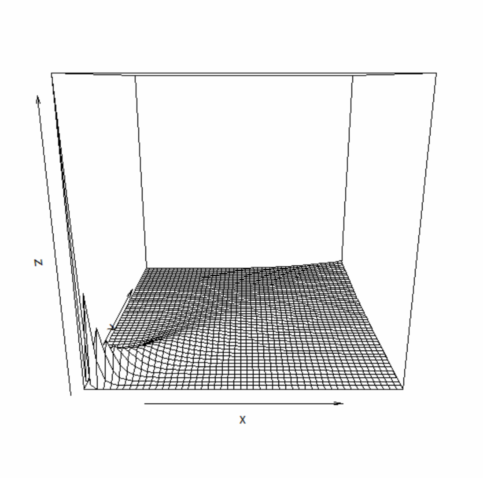
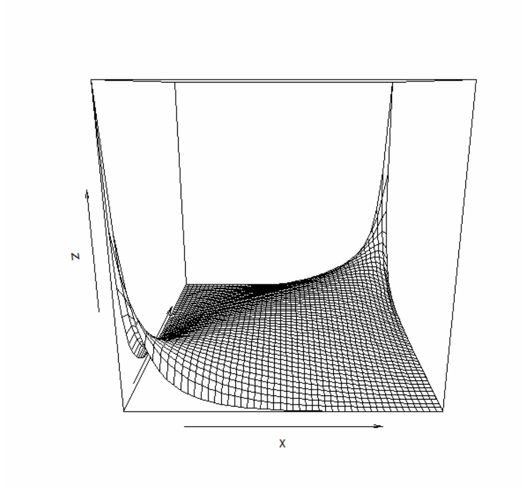


Figure 3.q Gauss Copula: INR-EURO and INR-GBP Figure 3.r   t-Copula: INR-EURO and INR-GBP



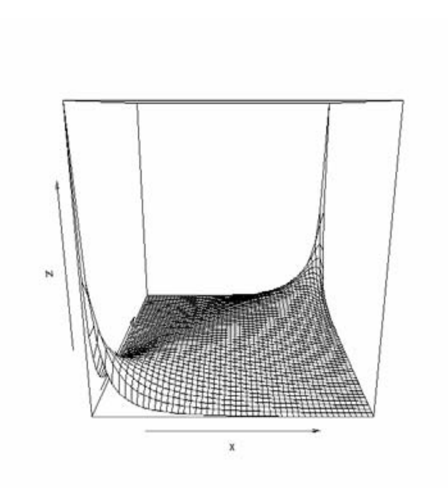
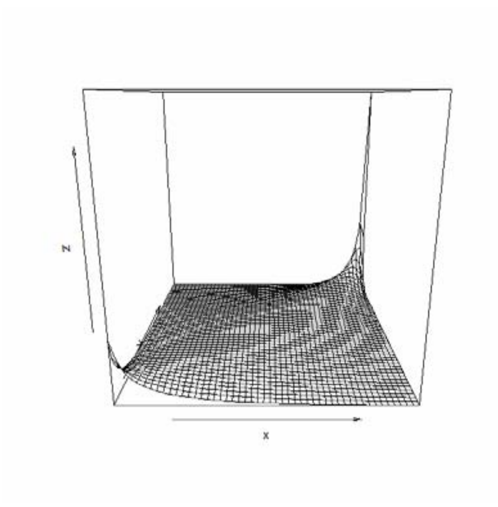


Figure 3.s Gumble -Copula: INR-EURO and INR-GBP Figure 3.t   Clayton -Copula: INR-EURO and INR-GBP



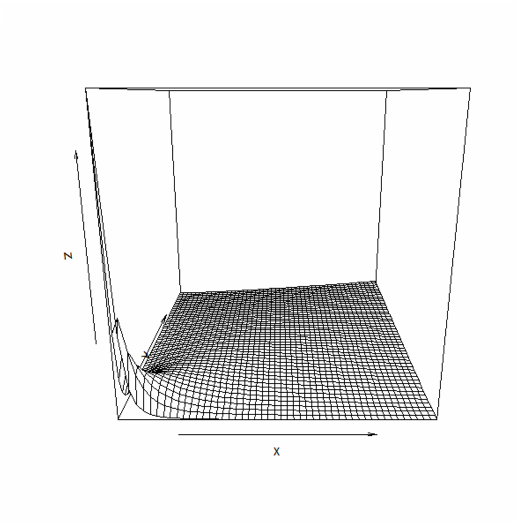
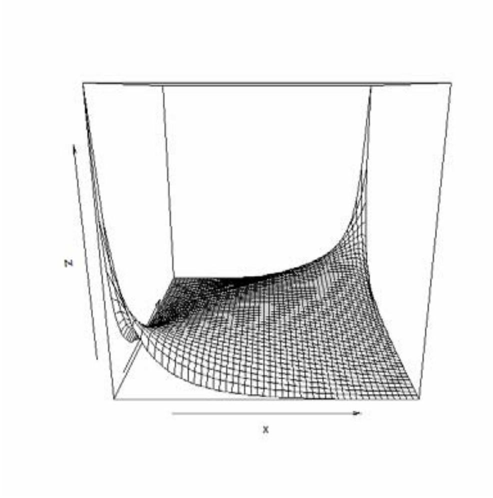


Figure 3.u Gauss Copula: INR-CHF and INR-GBP Figure 3.v t-Copula: INR-CHF and INR-GBP



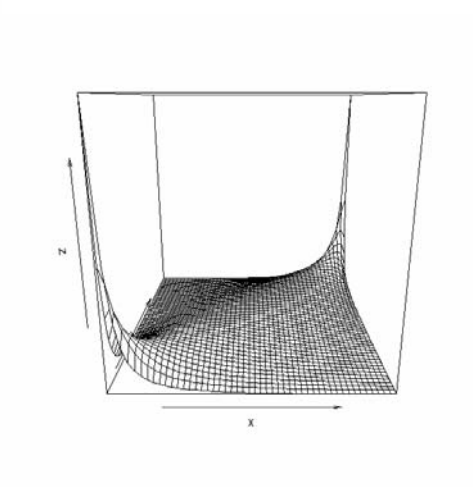
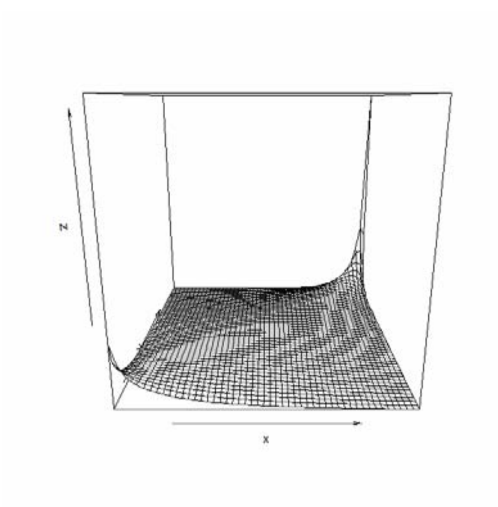
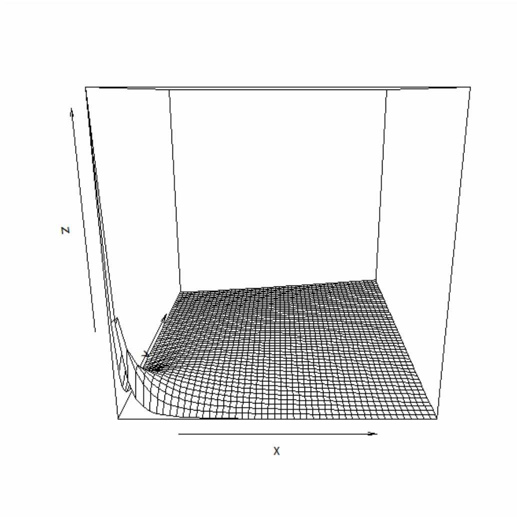


Figure 3.w Gumble -Copula: INR-CHF and INR-GBP Figure 3.x Clayton -Copula: INR-CHF and INR-GBP





8.3. Comparison of models

To compare the goodness of fit and performance of the models we use both the log likelihood values and back testing of results (in terms of number occasion when actual exceed the VaR number).

8.3.1 Log likelihood

Estimated parameters and log likelihood values (log likelihood values of copula models are positive and for non-copula models are negative) of all seven models are given in Annex V. Clayton copula and multivariate ‘t’ distribution exhibit best in-sample fit.

8.3.2 Back testing

For the purpose of back testing, we use the 200 days data and calculate the number of occurrence of exceptions under each model. Also we compute the Kupiec test and Christoferson test to check the effectiveness of the model. 99% VaR implies that out of 200 days two exceptions are acceptable. The back testing result shows that Model (i) produces 10 exceptions, model (ii), model (iii), model (v) and model (vi) produces three exceptions each and Model (iv) and model (vii) produces two exceptions each. However, Kupiec tests and Christoferson test indicates that for model (i) the null hypothesis is rejected which implies that the VaR estimate using model (i) is not accurate. For all other models null hypothesis cannot be rejected in both the tests.

9. Conclusion

In this paper, we use high-frequency multivariate data and attempt to model the joint distribution (dependency structure) of daily returns of four major foreign currencies against Indian rupees. Like in many previous works, the modelling framework we adopt here is mainly a copula-GARCH model. In particular, we use ARMA-GARCH specification to filter the deterministic terms in the daily return series and then model the residuals using various statistical techniques such as (i) multivariate normal distribution; (ii) multivariate t-distribution; (iii) converting the individual series so that transformed variables follows Normal distribution (Hull-White transformation) and thereafter fitting these variable to a multivariate normal-distribution; (iv) transforming the individual series so that transformed variables follows student's t distribution (Hull-White transformation) and thereafter fitting these variable to a multivariate t-distribution; (v) Gauss-copula; (vi) t-Copula; and (vii) Clayton-copula. Thereafter, we compute portfolio VaR using Monte Carlo simulation technique for the portfolio with four risk factors (INR-USD, INR-EURO, INR-GBP and INR-CHF exchange rates) of equal weights for each of the seven models of dependency structure. We also compare the performances of these models based on the log likelihood values of model fit to the data as well as back testing of VaR results. As part of back testing of VaR results, one-day portfolio VaRs were computed for the 200 days for the hypothetical portfolio which depends on these four risk factors using Monte Carlo Simulations technique, for each of the seven models of dependency structure. It is observed that multivariate normal distribution does not provide a good in-sample fit of the joint distribution of four exchange rate returns under consideration, and also number of exceptions raised in backtesting of VaR estimate are exceptionally high and also unconditional coverage test (binomial test/ kupiec test) and conditional coverage test (christoffersen test) suggest that the VaR estimate is inaccurate. In contrast, VaR estimate based on other six models produce acceptable VaR estimate. However, among these models, Clayton copula model (model vii) and multivariate student's t distribution after transforming individual exchange rate returns to student's t distribution (Hull-White transformation - model iv) produce least number of exceptions (2 out of 200 days) in back testing of VaR estimate.

References:

Alexander J. McNeil, Rudiger Frey, Paul Embrechts, 2005, Quantitative Risk Management: Concepts, Techniques and Tools; Princeton University Press, Princeton and Oxford

Ang, A. and J. Chen (2002). Asymmetric Correlations of Equity Portfolios, *Journal of Financial Economics* **63**, 443-494.

Bollerslev, T, 1986, "Generalized autoregressive conditional heteroskedasticity". Journal of Econometrics 31:307–327.

Bouy´e, E., Durrleman, V., Nikeghbali, A., Riboulet, G. and Roncalli, T. (2000). Copulas for finance, a reading guide and some applications. Working paper, Financial Econometrics Research Centre, City University, London.

Cherubini, U. and Luciano, E. (2001). Value at risk trade-off and capital allocation with copulas. *Economic Notes* **30**, 235-256.

Clayton, D. G. 1978. "A model for association in bivariate life tables and its application in epidemiological studies of familial tendency in chronic disease incidence.", Biometrika 65:141–151.

Engle, R. F, 1982, "Autoregressive conditional heteroskedasticity with estimates of the variance of UK inflation.", Econometrica 50:987–1008.

Engle, R. F. and T. Bollerslev,1986, "Modeling the persistence of conditional variances.", Econometric Review 5:1–50.

Embrechts, P. and Hoing, A., Juri, A. (2003). Using copulae to bound the value-at-risk for functions of dependent risks. *Finance and Stochastics* **7**, 145-167.

Embrechts, P., McNeil, A. & Straumann, D. (2001). "Correlation and dependency in risk management: properties and pitfalls. In Risk Management: Value at Risk and Beyond", Cambridge University Press, pp. 176–223.

Fortin, I. and Kuzmics, C. (2002). Tail dependence in stock return pairs. *International Journal of Intelligent Systems in Accounting, Finance & Management* **11**, 89-107.

Goeij, P., and Wessel Marquering, (2004), "Modeling the Conditional Covariance Between Stock and Bond Returns: A Multivariate GARCH Approach", Journal of Financial econometrics, Vol. 2, No. 4, 531-564.

Hamilton, J. 1994. "Time Series Analysis", Princeton University Press.

Hull, J. and A. White, 1998, “Value at Risk when Daily Changes in Market Variables are not Normally Distributed,” *Journal of Derivatives*, Spring 1998, pp 9-19.
----------1998. "Incorporating volatility updating into the historical simulation method for VaR", Journal of Risk 1(1):5–19.
---------- 2001. "Valuing credit default swaps. II. Modelling default correlations", Journal of Derivatives 8(3):12–22.
---------. 2004."Valuation of a CDO and an nth to default CDS without Monte Carlo simulation", Journal of Derivatives 12:8–23.

Joe, H. (1997). *Multivariate Models and Dependence Concepts*. Chapman and Hall.

Jorion, P. 2002. "Financial Risk Manager Handbook", 2001–2002.Wiley

Jose A. Lopez,1999, "Regulatory Evaluation of Value-at-Risk Models", Research Paper, Research and Market Analysis Group, Federal Reserve Bank of New York.

JPMorgan. 1996. "RiskMetrics Technical Documen", 3rd edn. NewYork: JPMorgan.

Kupiec, P.,1995,“Techniques for Verifying the Accuracy of Risk Measurement Models,” Journal of Derivatives, 3, 73-84.

Longin, F. and Solnik, B. (2001). Extreme correlation of international equity markets. *Journal of Finance* **56**, 649-676.
Long Kang (2007), “Modeling the Dependence Structure between Bonds and Stocks: A Multidimensional Copula Approach”, Indiana University Bloomington

L¨utkepohl, H. 1993, "Introduction to Multiple Time Series Analysis", 2nd edn. Springer.

McNeil, A. J and R. Frey, 2000, "Estimation of tail-related risk measures for heteroskedasticity financial time series: an extreme value approach" Journal of Empirical Finance 7:271–300.

Meneguzzo, D. and Vecchiato, W. (2002). Copulas sensitivity in collaterized debt obligations and basket defaults swaps pricing and risk monitoring. Working paper, Veneto Banca.

Nelsen, R. B,1998, "An Introduction to Copulas", Springer.

Patton, A. J. (2002). Modelling time-varying exchange rate dependence using the conditional copula. Working paper, UCSD.

Patton, A. (2003). Modelling asymmetric exchange rate dependence. Working paper, University of California, San Diego.

Patton, Andrew J. (2004), "On the Out-of-Sample Importance of Skewness and Asymmetric Dependence for Asset Allocation." Journal of Financial Econometrics, Vol. 2, No. 1, pp. 130-168.

Pravin K. Trivedi and David M. Zimmer, 2005, Copula Modeling: An Introduction for Practitioners; Foundations and Trends in Econometrics Vol. 1, No 1 (2005) 1–111

Sklar, A. (1959), "Fonctions de repartition an dimensions et leurs marges", Publications de l’Institut de Statistique de L.Universit de Paris 8, 229-231.

Tasfack, Georges (2006), "Dependence Structure and Extreme Comovements in International Equity and Bond Markets”, Working paper, Universite de Montreal, CIRANO and CIREQ.

Thomas J. Linsmeier and Neil D. Pearson, 1996, "Risk Measurement: An Introduction to Value at Risk"

Zivot, E. and J.Wang. 2003. "Modeling Financial Time Series with S-PLUS", Springer.

Examples of Copula

Product Copula/ Independence copula

The probability of multiple events, if they are independent, can be specified by multiplying individual probabilities together. *Two random variables R*1 *and R*2 *are independent if and only if the product of their distribution functions F*1 *and F*2 *equals their joint distribution function.* Hence the product copula is defined as

$C(u1, ..., un) = P(U1<u1, ..., Un<un) = P(U1<u1)\, P(U2<u2) ... P(Un<un) = u1\, u2\, .\, un.$

Gaussian copula

If we transform the random variable (X) using CDF to U which is uniformly distributed and transform them again so that they become normally distributed using the inverse Normal probability function. The joint distribution of these Normal variables is then assumed to be multivariate Normal, with a given correlation matrix $\tilde{\sum}$

Writing Φ_{Σ} for the probability P(*Z*1<*z*1, ..., *Zn*<*zn*), where the *Z*.s are multivariate Normal with correlation matrix ∑, we have

$C(u1, ..., un) = \Phi_{\Sigma}\, (\Phi^{-1}(u1), ..., \Phi^{-1}(un))$.

$$\Phi_{\Sigma}(z_1,\dots,z_n) = \int_{-\infty}^{z_1}\cdots\int_{-\infty}^{z_n} \frac{\exp\left(-\frac{1}{2}\sum_{i,j=1}^{n} z_i(\Sigma^{-1})_{ij} z_j\right)}{(2\pi)^{n/2}(\det\Sigma)^{1/2}} dz_1\cdots dz_n$$

When n=2, the bivariate gauss copula as proposed by Lee (1983) takes the form:

$$\begin{aligned} C(u_1,u_2;\theta) &= \Phi_G\left(\Phi^{-1}(u_1),\Phi^{-1}(u_2);\theta\right), \\ &= \int_{-\infty}^{\Phi^{-1}(u_1)}\int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi(1-\theta^2)^{1/2}} \\ &\quad \times \left\{\frac{-(s^2-2\theta st+t^2}{2(1-\theta^2)}\right\} dsdt \end{aligned}$$

Where Φ is the CDF of the standard normal distribution, and $\Phi_G(u_1,u_2)$ is the standard bivariate normal distribution with correlation parameter θ.

Student's t-copula

Similarly, the t copula is the unique copula of X

$C(u1, ..., un) = P(U1<u1, ..., Un<un)= T_{v,\Sigma}(\, T_v^{-1}(u_1),... T_v^{-1}(u_n))$

with T_v denoting the cumulative distribution function of the univariate Student-t distribution and $T_{v,\Sigma}$ denoting the cumulative distribution function of the multivariate Student-t distribution.

Gumbel Copula

Bivariate Gumbel Copula is given by the formula

$$C_\theta(u,v) \stackrel{\text{def}}{=} \exp\left\{-\left[(-\ln u)^\theta + (-\ln v)^\theta\right]^{1/\theta}\right\}$$

The parameter θ may take all values in the interval [1,∞). For *θ=1 the gumbel copula becomes product/independent copula.* The Gumbel copula interpolates between independence and perfect dependence and the parameter θ represents the strength of dependence. To calibrate Gumbel copula to empirical data sample correlation coefficient (Kendall's tau ρ_τ) is used and the formula is $\rho_\tau = 1 - 1/\theta$.

Clayton Copula

Bivariate Clayton Copula is given by the formula

$$C_\theta^{\mathrm{Cl}}(u_1,u_2) = (u_1^{-\theta} + u_2^{-\theta} - 1)^{-1/\theta}, \quad 0 < \theta < \infty$$

As θ --> 0, Clayton copula becomes independent copula. To calibrate Clayton copula to empirical data sample correlation coefficient (Kendall's tau ρ_τ) is used and the formula is: $\rho_\tau = \theta/(\theta+2)$.

The Student-t copula has positive tail dependence whenever the correlation is positive. The coefficients of upper and lower tail dependence are equal, by symmetry. The Clayton copula has positive lower tail dependence ($2^{-1/\theta}$) but no upper tail dependence. The Gumbel copula has positive upper tail dependence ($2-2^{-1/\theta}$), but no lower tail dependence.

Annex II

p-value of normality test of exchange rate returns

	Anderson-Darling	Cramer-von Mises	Kolmogorov-Smirnov	Shapiro-Francia	Pearson chi-square
INR-USD	< 2.2e-16	< 2.2e-16	< 2.2e-16	< 2.2e-16	< 2.2e-16
INR-EUR	< 2.2e-16	4.043e-10	3.366e-10	< 2.2e-16	1.845e-08
INR-CHF	< 2.2e-16	4.635e-10	1.88e-10	< 2.2e-16	1.681e-06
INR-GBP	< 2.2e-16	3.76e-05	1.204e-14	< 2.2e-16	4.544e-12

Equation 1: GARCH model for INR-CHF daily exchange rate return

Dependent Variable: CHF
Method: ML - ARCH (Marquardt) - Normal distribution
Included observations: 2739 after adjustments
Convergence achieved after 9 iterations
MA Backcast: 12/31/1999
Presample variance: backcast (parameter = 0.7)
GARCH = C(3) + C(4)*RESID(-1)^2 + C(5)*GARCH(-1)

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	0.000116	0.000120	0.969923	0.3321
MA(1)	-0.071346	0.019630	-3.634583	0.0003
Variance Equation				
C	7.56E-07	2.23E-07	3.385439	0.0007
RESID(-1)^2	0.045126	0.006588	6.849306	0.0000
GARCH(-1)	0.941207	0.009017	104.3812	0.0000

R-squared	0.007395	Mean dependent var	0.000182
Adjusted R-squared	0.007032	S.D. dependent var	0.007567
S.E. of regression	0.007541	Akaike info criterion	-7.038117
Sum squared resid	0.155627	Schwarz criterion	-7.027318
Log likelihood	9643.701	Hannan-Quinn criter.	-7.034214
F-statistic	5.097607	Durbin-Watson stat	2.032422
Prob(F-statistic)	0.000431		
Inverted MA Roots	.07		

Equation 2: GARCH model for INR-EURO daily exchange rate return

Dependent Variable: EUR
Method: ML - ARCH (Marquardt) - Normal distribution
Date: 12/17/10 Time: 10:35
Sample (adjusted): 1/04/2000 7/01/2010
Included observations: 2738 after adjustments
Convergence achieved after 10 iterations
Presample variance: backcast (parameter = 0.7)
GARCH = C(3) + C(4)*RESID(-1)^2 + C(5)*GARCH(-1)

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	0.000120	0.000113	1.064834	0.2870
AR(1)	-0.054559	0.019895	-2.742354	0.0061
Variance Equation				
C	7.20E-07	1.70E-07	4.228204	0.0000
RESID(-1)^2	0.051752	0.006489	7.975916	0.0000
GARCH(-1)	0.933656	0.008122	114.9531	0.0000

R-squared	0.003560	Mean dependent var	0.000117
Adjusted R-squared	0.003196	S.D. dependent var	0.006943
			-
S.E. of regression	0.006932	Akaike info criterion	7.207689
			-
Sum squared resid	0.131470	Schwarz criterion	7.196887
			-
Log likelihood	9872.326	Hannan-Quinn criter.	7.203786
F-statistic	2.443586	Durbin-Watson stat	2.016019
Prob(F-statistic)	0.044661		

Inverted AR Roots	-.05

Equation 3: GARCH model for INR-GBP daily exchange rate return

Dependent Variable: GBP
Method: ML - ARCH (Marquardt) - Normal distribution
Convergence achieved after 13 iterations
MA Backcast: 12/31/1999
Presample variance: backcast (parameter = 0.7)
GARCH = C(3) + C(4)*RESID(-1)^2 + C(5)*GARCH(-1)

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	7.65E-05	0.000103	0.739594	0.4595
MA(1)	-0.061719	0.019457	-3.172031	0.0015
		Variance Equation		
C	5.96E-07	1.22E-07	4.881099	0.0000
RESID(-1)^2	0.052704	0.005890	8.948781	0.0000
GARCH(-1)	0.933170	0.007298	127.8618	0.0000

R-squared	0.004188	Mean dependent var	6.98E-06
Adjusted R-squared	0.003824	S.D. dependent var	0.006665
			-
S.E. of regression	0.006652	Akaike info criterion	7.368236
			-
Sum squared resid	0.121121	Schwarz criterion	7.357437
			-
Log likelihood	10095.80	Hannan-Quinn criter.	7.364334
F-statistic	2.877372	Durbin-Watson stat	2.007843
Prob(F-statistic)	0.021580		

Inverted MA Roots	.06

Equation 4: GARCH model for INR-USD daily exchange rate return

Dependent Variable: USD
Method: ML - ARCH (Marquardt) - Normal distribution
Included observations: 2739 after adjustments
Convergence achieved after 6 iterations
MA Backcast: 12/31/1999
Presample variance: backcast (parameter = 0.7)
GARCH = C(3) + C(4)*GARCH(-1)

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	-3.10E-05	6.94E-05	-0.447228	0.6547
MA(1)	-0.094654	0.009674	-9.783973	0.0000
		Variance Equation		
C	2.20E-06	1.82E-06	1.207169	0.2274
GARCH(-1)	0.862185	0.114189	7.550513	0.0000

R-squared	0.009786	Mean dependent var	1.40E-05
Adjusted R-squared	0.009424	S.D. dependent var	0.004019
S.E. of regression	0.004000	Akaike info criterion	-8.206186
Sum squared resid	0.043789	Schwarz criterion	-8.197547
Log likelihood	11242.37	Hannan-Quinn criter.	-8.203064
F-statistic	9.016102	Durbin-Watson stat	2.010962
Prob(F-statistic)	0.000006		

Inverted MA Roots	.09

Annex IV

'R' Script to estimate the parameter of the multivariate distributions, Monte Carlo simulation for VaR computation and back testing the results (**'R' version 2.12.0**)

```
library(QRMlib)
########## reading the data file from a .txt file which contains four series of daily returns of exchange rates and four series of corresponding innovation series (using GARCH filter) ####
exch<-read.table(file="e:\\copula\\exch.txt",header=T)
#############assigning the daily return series
usd<-exch$usd
chf<-exch$chf
eur<-exch$eur
gbp<-exch$gbp
############### assigning the innovation i.e. standradised (GARCH) return series
usd1<-exch$usd1
chf1<-exch$chf1
eur1<-exch$eur1
gbp1<-exch$gbp1
############### variables and matrices initialisation
i=j=k=0
a<-matrix(0,200,2500)
b<-matrix(0,200,2500)
c<-matrix(0,200,2500)
d<-matrix(0,200,2500)
dd<-matrix(0,200,2500)
rmn1<-matrix(0,2500,1)
rnq=matrix(0,200,1)
rtq=matrix(0,200,1)
rgq=matrix(0,200,1)
rtcq=matrix(0,200,1)
racq=matrix(0,200,1)
rhwtq=matrix(0,200,1)
rhwnq=matrix(0,200,1)
####Model calibration and Back testing: using 2500 observations for model calibration and 200 observations for back testing ###
for (i in 1:200){
j=1+i;k=2500+i;
a[i,]=usd1[j:k];
b[i,]=eur1[j:k];
c[i,]=chf1[j:k];
dd[i,]=gbp1[j:k];
d=cbind(a[i,],b[i,],c[i,],dd[i,]);
############converting the original series to uniform distribution by way of cumulative density function
 U <- apply(d,2,edf,adjust=1);
```

```
#### univariate t dist
t1<-fit.st(d[,1])
t2<-fit.st(d[,2])
t3<-fit.st(d[,3])
t4<-fit.st(d[,4])
#####################  Hull-white transformation - t dist
h1<-ecdf(d[,1])
hw1<-2499/2500*h1(d[,1])
h2<-ecdf(d[,2])
hw2<-2499/2500*h2(d[,2])
h3<-ecdf(d[,3])
hw3<-2499/2500*h3(d[,3])
h4<-ecdf(d[,4])
hw4<-2499/2500*h4(d[,4])
hwt1<-qt(hw1,df=t1$par.est[1])
hwt2<-qt(hw2,df=t2$par.est[1])
hwt3<-qt(hw3,df=t3$par.est[1])
hwt4<-qt(hw4,df=t4$par.est[1])
hwt<-cbind(hwt1,hwt2,hwt3,hwt4)
fhwt<-fit.mst(hwt)
rmhwt<-rmvt(n=1000, sigma = fhwt$Sigma, df = fhwt$nu)
x1<-pt(rmhwt[,1],df=t1$par.est[1])
x2<-pt(rmhwt[,2],df=t2$par.est[1])
x3<-pt(rmhwt[,3],df=t3$par.est[1])
x4<-pt(rmhwt[,4],df=t4$par.est[1])
y1<-quantile(d[,1],x1)
y2<-quantile(d[,2],x2)
y3<-quantile(d[,3],x3)
y4<-quantile(d[,4],x4)
rmhwt1<-y1+y2+y3+y4
rhwtq[i]<-quantile(rmhwt1,.01)
### Hull-White transformation –normal distribution
hwn1<-qnorm(hw1)
hwn2<-qnorm(hw2)
hwn3<-qnorm(hw3)
hwn4<-qnorm(hw4)
hwn<-cbind(hwn1,hwn2,hwn3,hwn4)
fhwn<-fit.norm(hwn)
rmhwn<-rmnorm(n=1000, fhwn$Sigma,fhwn$mu)
x5<-pnorm(rmhwn[,1])
x6<-pnorm(rmhwn[,2])
x7<-pnorm(rmhwn[,3])
x8<-pnorm(rmhwn[,4])
```

```
y5<-quantile(d[,1],x5)
y6<-quantile(d[,2],x6)
y7<-quantile(d[,3],x7)
y8<-quantile(d[,4],x8)
rmhwn1<-y5+y6+y7+y8
rhwnq[i]<-quantile(rmhwn1,.01)
##########  multivariate  normal distribution
fn<-fit.norm(d)
rmn<-rmnorm(1000,fn$Sigma,fn$mu);
rmn1<-rmn[,1]+rmn[,2]+rmn[,3]+rmn[,4]
rnq[i]<-quantile(rmn1,.05)
##########  multivariate t distribution
ft<-fit.mst(d)
rmt<-rmvt(n=1000, sigma = ft$Sigma, df = ft$nu)
rmt1<-rmt[,1]+rmt[,2]+rmt[,3]+rmt[,4]
rtq[i]<-quantile(rmt1,.01)
##########  multivariate: Gauss-Copula
mod.gauss <- fit.gausscopula(U);
rgc <- rcopula.gauss(1000,d=4,Sigma=mod.gauss$P);
rmgc1<-quantile(a[i,],rgc[,1])+quantile(b[i,],rgc[,2])+quantile(c[i,],rgc[,3])+quantile(dd[i,],rgc[,4])
rgq[i]<-quantile(rmgc1,.01)
##########  multivariate: t Copula
mod.t <- fit.tcopula(U);
rtc <- rcopula.t(1000,d=4,Sigma=mod.t$P,df=mod.t$nu);
rmtc1<-quantile(a[i,],rtc[,1])+quantile(b[i,],rtc[,2])+quantile(c[i,],rtc[,3])+quantile(dd[i,],rtc[,4])
rtcq[i]<-quantile(rmtc1,.01)
##########  multivariate: Clayton Copula
mod.g<- fit.AC(U,name="clayton");
rac<-rAC("clayton", n=1000, d=4,theta =mod.g$theta);
rmac1<-quantile(a[i,],rac[,1])+quantile(b[i,],rac[,2])+quantile(c[i,],rac[,3])+quantile(dd[i,],rac[,4])
racq[i]<-quantile(rmac1,.01)
}

########To plot the kernel densities of the daily exchange rate returns ########
 plot(density(usd),main="Kernel density of INR-USD exchange rate")
plot(density(eur),main="Kernel density of INR-EURO exchange rate")
plot(density(usd),main="Kernel density of INR-USD exchange rate")
plot(density(gbp),main="Kernel density of INR-GB.Pound exchange rate")
plot(density(chf),main="Kernel density of INR-CHF (swiss franc) exchange rate")
plot(density(rnorm(2738)),main="Kernel density of a typical standard normal variable")
```

1. Estimation of model I (multivariate normal distribution)

$\hat{\Sigma}$

	usd	chf	eur	gbp
usd	1.067			
chf	0.356	0.987		
eur	0.378	0.908	0.999	
gbp	0.374	0.734	0.684	0.999

$\hat{\rho}$

	usd	chf	eur	gbp
usd	1			
chf	0.347	1		
eur	0.366	0.915	1	
gbp	0.363	0.74	0.685	1

$\hat{\mu}$:

usd	chf	eur	gbp
-0.0033	0.028	0.03	0.011

Log likelihood: -10780.55

2. Estimation of model II (multivariate t-distribution)

$\hat{\Sigma}$

	usd	chf	eur	gbp
usd	0.404			
chf	0.135	0.635		
eur	0.137	0.597	0.645	
gbp	0.147	0.463	0.448	0.633

$\hat{\rho}$

	usd	chf	eur	gbp
usd	1			
chf	0.267	1		
eur	0.268	0.933	1	
gbp	0.292	0.731	0.701	1

$\hat{\mu}$:

usd	chf	eur	gbp
-0.0172	0.025	0.015	0.024

$\hat{\nu}$: 4.114553

Log likelihood: -9987.231

3. Estimation of Model III

$\hat{\Sigma}$

	usd	chf	eur	gbp
usd	0.9962			
chf	0.2988	0.9955		
eur	0.3178	0.9103	0.9953	
gbp	0.3275	0.7330	0.6814	0.9956

$\hat{\rho}$

	usd	chf	eur	gbp
usd	1.0000			
chf	0.3001	1.0000		
eur	0.3192	0.9145	1.0000	
gbp	0.3289	0.7363	0.6845	1.0000

$\hat{\mu}$:

usd	chf	eur	gbp
0.0114	0.0016	0.0017	0.0015

Log likelihood: -10756.55

4. Estimation of Model IV.

$\hat{\Sigma}$

	usd	chf	eur	gbp
usd	2.5954			
chf	0.3328	0.7538		
eur	0.3293	0.6968	0.7394	
gbp	0.3781	0.5599	0.5337	0.7865

$\hat{\rho}$

	usd	chf	eur	gbp
usd	1			
chf	0.238	1		
eur	0.238	0.933	1	
gbp	0.265	0.727	0.7	1

$\hat{\nu}$: 3.231079

$\hat{\mu}$:

usd	chf	eur	gbp
0.0037	-0.0109	-0.0191	0.0045

Log likelihood: -13588.47

5. Estimation of Model V (Normal Copula)

$\hat{P}$

====================================

	usd1	chf1	eur1	gbp1
usd1	1			
chf1	0.3015	1.0000		
eur1	0.3205	0.9149	1.0000	
gbp1	0.3302	0.7374	0.6858	1

Log likelihood 3409.397

6. Estimation of Model VI (Student-t copula)

$\hat{P}$

====================================

	usd1	chf1	eur1	gbp1
usd1	1			
chf1	0.2832	1		
eur1	0.2873	0.9299	1	
gbp1	0.3148	0.7332	0.7000	1

ν: 5.807568

Log likelihood: 3694.966

7. Estimation of Model VII (Clayton copula)

Theta: 0.8030917

Log likelihood: 1484.095