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Abstract

Recently introduced C-divergence estimators as well as the associated test statistics have shown a good robustness behavior. However, one shortcoming of these test statistics is that their asymptotic distribution, in general, is not a chi-square distribution but a linear combination of chi-square distributions. In this paper, therefore, we consider Wald-type test statistics based on minimum C-divergence estimators to overcome this shortcoming. We establish that this family of test statistics is a chi-square distribution and compute an approximation of the power function under simple null hypothesis and composite null hypothesis. We calculate both first order and second order influence function of the Wald-type test statistics, based on which the robustness of the family of test statistics can be inferred. Both simulated and real data examples have been shown as part of numerical results.

JEL Classification: C12, C13, C15, C18

Keywords: C-divergence, Robust Statistics, Wald-type Test

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Robust Wald-type Test Statistics based on Minimum C-divergence Estimators

1. Introduction

In the last few years many papers have been published based on minimization of a suitable statistical distance or divergence in order to present robust estimators in parametric inference as well as robust parametric tests based on them. Many of the statistical distances or divergences are members of the ϕ-divergences or the Bregman's distance. In the first case, the most important subfamily is the Cressie and Read (CR) family of divergence measures (Pardo, 2006) while in the second case *i.e.*, Bregman distances, the most important family is the density power divergence (DPD) considered for the first time in Basu *et al.* (1998). Based on these two divergence families (CR and DPD), the C-divergence family was introduced as a generalization of the CR divergence and the DPD (Maji *et al.*, 2018). The minimum C-divergence estimators (MCDEs), based on C-divergence family, manifested substantially superior performance, compared to likelihood ratio test, especially in the presence of outliers. On the other hand, without outliers the test statistics were found to be competitive, in general, to the likelihood ratio test. Therefore, the estimators as well as test statistics based on C-divergence measures can serve as useful practical tools in robust statistics.

The test statistics based on C-divergence measures, considered in Maji *et al.* (2018), however, have two small shortcomings. The first is in relation to get the C-divergence measure between two populations in a concrete family of probability distributions. The functional form usually does not boil down to a simpler form and we need to go for numerical methods to evaluate them. While the second one is in relation to the asymptotic distribution of the above test statistics, as in many situations, it is a linear combination of independent chi-squared random variables instead of a chi-squared distribution. It is true that in this moment there are many procedures that can be used while working with linear combination of chi-squared random variables but it is more useful to have a chi-squared distribution as the asymptotic distribution of the test statistics. For similar use of the single chi-squared distribution instead of the linear combination of the chi-squared distributions, see Basu *et al.* (2016). To overcome the above mentioned issues, we introduce Wald-type tests based on MCDEs in this paper. The architecture of this paper is in similar lines of Basu *et al.* (2016), but the motivation of the paper differentiates it as well. We show that in this case the asymptotic distribution is a chi-squared distribution and further show that the Wald-type tests based on MCDEs show robust properties. Rest of the paper is organized as follows. Section 2 presents some results obtained in Maji *et al.* (2018). The Wald-type test statistics based on the MCDEs are analysed in Section 3. The asymptotic distribution is obtained for the simple as well as for the composite null hypothesis in Section 3. Some approximations to the power function are also prescribed in the same section. The influence function of the Wald-type tests based on the MCDEs are obtained in Section 4. A simulation study is carried out in Section 5. Some real data examples are considered in Section 6. A small discussion about the tuning parameter appearing in the Wald-type tests is developed in Section 7. Concluding remarks are in Section 8.

2. Background: The C-Divergence

In this section, we pay special attention to the definition of the C-divergence measure, the MCDEs as well as the asymptotic distribution of the MCDEs.

2.1 *C-Divergence Measure and the MCDEs*

Let $\mathbb{G}$ denote the family of all distributions having densities with respect to the Lebesgue measure or the counting measure. Given two densities g and f in $\mathbb{G}$, the C-divergence measure is defined as:

$$C(g,f) = \int N\left(\frac{g(x)}{f(x)} - 1\right) f^{1+\alpha}(x)\, dx \tag{1}$$

where $\frac{g}{f} - 1 = \upsilon$ is the Pearson residual, $\alpha \in (-\infty, \infty)$ is the tuning parameter and $N(\upsilon)$ is a function satisfying following properties. $N(\cdot)$ is thrice differentiable and strictly convex on $[-1, \infty)$ with $N(0) = 0$ and $N'(0) = 0$. The function $N(\cdot)$ itself may depend on one or more tuning parameters. Note that the recently developed S-divergence family of Ghosh *et al.* (2017) becomes a particular subfamily of our general C-divergence family where

$$N(\upsilon) = N_{\alpha,\lambda}(\upsilon) = \frac{1}{A} - \frac{1+\alpha}{AB}(1+\upsilon)^{A} + \frac{1}{B}(1+\upsilon)^{1+\alpha}, \quad \alpha \geq 0, \lambda \in \mathbb{R} \tag{2}$$

with $A = 1 + \lambda(1-\alpha)$ and $B = \alpha - \lambda(1-\alpha)$. Note that $N(0) = 0$ by default.

One prominent new member of the C-divergence family can be obtained by choosing $N(\upsilon)$ as the function $\xi_\lambda(\upsilon)$ where $\xi_\lambda(\upsilon) = \frac{(\upsilon+1)^{\lambda+1} - (\upsilon+1)}{\lambda(\lambda+1)} - \frac{\upsilon}{\lambda+1}$. The resulting subfamily, which we refer to as the Generalized Power Divergence ($\mathrm{GPD}_{\alpha,\lambda}$) family, has the form:

$$\mathrm{GPD}_{\alpha,\lambda}(g,f) = \int \xi_\lambda(x) f^{1+\alpha}(x)\, dx \tag{3}$$

Note that, by substituting $\alpha = 0$ in Equation (3), one gets the ordinary power divergence family.

$$\begin{aligned} \mathrm{GPD}_{\alpha=0,\lambda}(g,f) &= \int \xi_\lambda(x) f(x)\, dx \\ &= \int \left[\frac{1}{\lambda(\lambda+1)}\left\{g(x)\left[\left(\frac{g(x)}{f(x)}\right)^{\lambda} - 1\right]\right\} - \frac{g(x)-f(x)}{\lambda+1}\right] dx \end{aligned} \tag{4}$$

As a particular member of the $\mathrm{GPD}_{\alpha,\lambda}(g,f)$ family and keeping $\alpha = \lambda$ gives a scaled version of the DPD family:

$$\mathrm{GPD}_{\alpha,\alpha}(g,f)=\int\left\{f(x)^{1+\alpha}-\left(1+\frac{1}{\alpha}\right)g(x)f(x)^{\alpha}+\frac{1}{\alpha}g(x)^{1+\alpha}\right\}\,dx,\alpha>0 \tag{5}$$

with tuning parameter α. It can be seen that

$$\lim_{\lambda\to 0}\mathrm{GPD}_{\alpha=0,\lambda}(g,f)=\lim_{\alpha\to 0}\mathrm{GPD}_{\alpha,\alpha}(g,f)=\int g(x)\log\frac{g(x)}{f(x)}\,dx \tag{6}$$

i.e., the Kullback-Leibler divergence between the density functions g and f. For more details about

$$\mathrm{GPD}_{\alpha=0,\lambda}(g,f)=\mathrm{CR}(g,f)$$

and $\mathrm{GPD}_{\alpha,\alpha}(g,f)=\mathrm{DPD}(g,f)$, see Pardo (2006) and Basu *et al.* (2011) respectively.

Let us consider a random sample $X_1, X_2, \ldots, X_n$ from the true density g which we model by the parametric family $\mathbb{F}=\{f_{\boldsymbol{\theta}}:\boldsymbol{\theta}\in\Theta\subseteq\mathbb{R}^p\}$. We are interested in estimating the parameter $\boldsymbol{\theta}$. The minimum C-divergence functional $T_\alpha(G)$ at G is defined as:

$$C(g,f_{T_\alpha(G)})=\min_{\boldsymbol{\theta}\in\Theta}C(g,f_{\boldsymbol{\theta}})$$

where $C(\cdot,\cdot)$ is as defined in Equation (1).

Next, in order to estimate $\boldsymbol{\theta}$ based on the observed data, we have to minimize $C(\hat{g}_n, f_{\boldsymbol{\theta}})$ with respect to $\boldsymbol{\theta}$, where $\hat{g}_n$ is the vector of relative frequencies or some continuous density estimate based on the data according to whether the set-up is discrete or continuous. The estimating equation is then given by:

$$-\int\{(1+\alpha)N(\upsilon_n(x))-N'(\upsilon_n(x))(\upsilon_n(x)+1)\}f_{\boldsymbol{\theta}}^{1+\alpha}(x)u_{\boldsymbol{\theta}}(x)\,dx=\mathbf{0}_p \tag{7}$$

where $\upsilon_n(x)=\frac{\hat{g}_n(x)}{f_{\boldsymbol{\theta}}(x)}-1$ and $u_{\boldsymbol{\theta}}(x)=\frac{\partial}{\partial\boldsymbol{\theta}}\log f_{\boldsymbol{\theta}}(x)$ is the likelihood score function. For simplicity, we rewrite Equation (7) as $\int K(\upsilon_n(x))f_{\boldsymbol{\theta}}^{1+\alpha}(x)u_{\boldsymbol{\theta}}(x)\,dx=\mathbf{0}_p$ with

$$K(\upsilon)=N'(\upsilon)(\upsilon+1)-(1+\alpha)N(\upsilon). \tag{8}$$

2.2 *Asymptotic Properties of the Minimum C-Divergence Estimators*

Let $\hat{\boldsymbol{\theta}}_{N,\alpha}$ be the MCDEs based on the random sample $X_1, X_2, \ldots, X_n$. Maji *et al.* (2018) have shown that under standard set of assumptions for both discrete and continuous models (for $g=f_{\boldsymbol{\theta}}$)

$$\sqrt{n}(\hat{\boldsymbol{\theta}}_{N,\alpha}-\boldsymbol{\theta})\xrightarrow[n\to\infty]{L}N_p(\mathbf{0}_p,\boldsymbol{\Sigma}_\alpha(\boldsymbol{\theta})) \tag{9}$$

where

$$\boldsymbol{\Sigma}_\alpha(\boldsymbol{\theta})=\boldsymbol{J}_\alpha^{-1}(\boldsymbol{\theta})\boldsymbol{V}_\alpha(\boldsymbol{\theta})\boldsymbol{J}_\alpha^{-1}(\boldsymbol{\theta}) \tag{10}$$

N_p denotes a normal distribution of order p, $\boldsymbol{J}_\alpha(\boldsymbol{\theta})$ and $\boldsymbol{V}_\alpha(\boldsymbol{\theta})$ are defined for both discrete and continuous models separately in Maji *et al.* (2018). We have the following forms of $\boldsymbol{J}_\alpha(\boldsymbol{\theta})$ and $\boldsymbol{V}_\alpha(\boldsymbol{\theta})$ for the discrete models:

$$\boldsymbol{J}_\alpha(\boldsymbol{\theta}) = \sum u_{\boldsymbol{\theta}}(x) u_{\boldsymbol{\theta}}^T(x) f_{\boldsymbol{\theta}}^{1+\alpha}(x); \tag{11}$$

$$\boldsymbol{V}_\alpha(\boldsymbol{\theta}) = \sum u_{\boldsymbol{\theta}}(x) u_{\boldsymbol{\theta}}^T(x) f_{\boldsymbol{\theta}}^{1+2\alpha}(x) - \boldsymbol{\iota}\boldsymbol{\iota}^T; \tag{12}$$

$$\boldsymbol{\iota} = \sum u_{\boldsymbol{\theta}}(x) f_{\boldsymbol{\theta}}^{1+\alpha}(x). \tag{13}$$

The forms of $\boldsymbol{J}_\alpha(\boldsymbol{\theta})$ and $\boldsymbol{V}_\alpha(\boldsymbol{\theta})$ for the continuous models are as follows:

$$\boldsymbol{J}_\alpha(\boldsymbol{\theta}) = \int \widetilde{u_{\boldsymbol{\theta}}}(x) \widetilde{u_{\boldsymbol{\theta}}}(x)^T \{f_{\boldsymbol{\theta}}^*(x)\}^{1+\alpha} dx; \tag{14}$$

$$\boldsymbol{V}_\alpha(\boldsymbol{\theta}) = \mathrm{V}ar \left[\int W(x, X, h) (f_{\boldsymbol{\theta}}^*(x))^\alpha \widetilde{u_{\boldsymbol{\theta}}}(x) dx \right] \tag{15}$$

where $f_{\boldsymbol{\theta}}^*(x) = \int W(x, y, h_n)\ dF_{\boldsymbol{\theta}}(y)$, $\widetilde{u_{\boldsymbol{\theta}}}(x) = \frac{\partial}{\partial \boldsymbol{\theta}} \log f_{\boldsymbol{\theta}}^*(x)$, the kernel W is defined in terms of a symmetric nonnegative density function $w(\cdot)$ as:

$$W(x, X_i, h_n) = \frac{1}{h_n}\ w\left(\frac{x - X_i}{h_n}\right),$$

where $W(x, y, h_n)$ is a smooth kernel function with bandwidth h_n, ∇ denotes the gradient with respect to $\boldsymbol{\theta}$ and $F_{\boldsymbol{\theta}}$ is the distribution function corresponding to $f_{\boldsymbol{\theta}}$.

3. Wald-type Tests Based on the MCDE: Definition and Asymptotic Distribution

In the last few years it has been very common in the statistical literature to consider Wald tests based on the minimum distance estimators instead of the maximum likelihood estimator. The resulting tests have an excellent behavior in relation to robustness with minimal loss of efficiency (Basu *et al.*, 2016, 2017, 2018 and Ghosh *et al.* 2016). Based on the results presented in Section 2 we introduce Wald-type test statistics based on the MCDEs in order to test simple and composite null hypothesis.

3.1 *The Simple Null Hypothesis*

We define a family of Wald-type test statistics based on the MCDEs for testing the null hypothesis

$$H_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0 \quad \text{against} \quad H_1 : \boldsymbol{\theta} \neq \boldsymbol{\theta}_0. \tag{16}$$

Definition 1. *Let* $\hat{\boldsymbol{\theta}}_{N,\alpha}$ *be the MCDE of* $\boldsymbol{\theta}$*. The family of the Wald-type test statistics* *for testing the null hypothesis in Equation (16) is given by:*

$$W_n^C(\boldsymbol{\theta}_0) = n(\hat{\boldsymbol{\theta}}_{N,\alpha} - \boldsymbol{\theta}_0)^T \boldsymbol{\Sigma}_\alpha^{-1}(\boldsymbol{\theta}_0)(\hat{\boldsymbol{\theta}}_{N,\alpha} - \boldsymbol{\theta}_0) \tag{17}$$

where $\boldsymbol{\Sigma}_\alpha(\boldsymbol{\theta}_0)$ is as defined in Equation (10).

Theorem 1. *The asymptotic null distribution of the proposed Wald-type test statistics, given in Equation (17), is a chi-squared (χ^2) distribution with p degrees of freedom.*

Proof: *Under the null hypothesis H_0 defined in equation (16), $\sqrt{n}(\hat{\boldsymbol{\theta}}_{N,\alpha} - \boldsymbol{\theta}_0) \xrightarrow[n\to\infty]{L} N_p(\mathbf{0}, \boldsymbol{\Sigma}_\alpha(\boldsymbol{\theta}_0))$ as p be the dimension of $\boldsymbol{\theta}$ and using standard properties for obtaining the asymptotic distribution of a quadratic form, we may show that the Wald-type test statistics defined in Equation (17) has a χ^2 distribution with p degrees of freedom. By $\xrightarrow{L}$ we are denoting convergence in law.*

Based on the previous result we reject H_0 given in equation (16) if

$$W_n^C(\boldsymbol{\theta}_0) > \chi^2_{p,\beta} \tag{18}$$

where $\chi^2_{p,\beta}$ be the $(1-\beta)$-th quantile of χ^2 distribution with p degrees of freedom.

We are going to give a result that will be important in order to get an approximation of the power function of the Wald-type test statistics given in Expression (18).

Theorem 2. *Let $\boldsymbol{\theta}^*$ be the true value of parameter with $\boldsymbol{\theta}^* \neq \boldsymbol{\theta}_0$. Then the convergence*

$$\sqrt{n}\left(l\left(\hat{\boldsymbol{\theta}}_{N,\alpha}\right) - l\left(\boldsymbol{\theta}^*\right)\right) \xrightarrow[n\to\infty]{L} N_p(\mathbf{0}, \sigma_W^2(\boldsymbol{\theta}^*)) \tag{19}$$

holds, where

$$l(\boldsymbol{\theta}) = (\boldsymbol{\theta} - \boldsymbol{\theta}_0)^T \boldsymbol{\Sigma}_\alpha^{-1}(\boldsymbol{\theta})(\boldsymbol{\theta} - \boldsymbol{\theta}_0) \tag{20}$$

and

$$\sigma_W^2(\boldsymbol{\theta}^*) = 4(\boldsymbol{\theta}^* - \boldsymbol{\theta}_0)^T \boldsymbol{\Sigma}_\alpha^{-1}(\boldsymbol{\theta}^*)(\boldsymbol{\theta}^* - \boldsymbol{\theta}_0). \tag{21}$$

Proof: *A first order Taylor series expansion of $l(\boldsymbol{\theta})$ at $\hat{\boldsymbol{\theta}}_{N,\alpha}$ around $\boldsymbol{\theta}^*$ gives*

$$l(\hat{\boldsymbol{\theta}}_{N,\alpha}) - l(\boldsymbol{\theta}^*) = \left(\frac{\partial l(\boldsymbol{\theta})}{\partial \theta}\right)_{\boldsymbol{\theta}=\boldsymbol{\theta}^*} (\hat{\boldsymbol{\theta}}_{N,\alpha} - \boldsymbol{\theta}^*) + o_p(||\hat{\boldsymbol{\theta}}_{N,\alpha} - \boldsymbol{\theta}^*||). \tag{22}$$

The asymptotic distribution of $\sqrt{n}\left(l\left(\hat{\boldsymbol{\theta}}_{N,\alpha}\right) - l\left(\boldsymbol{\theta}^*\right)\right)$ coincides with the asymptotic distribution of

$$\sqrt{n}\left(\frac{\partial l(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right)_{\boldsymbol{\theta}=\boldsymbol{\theta}^*} (\hat{\boldsymbol{\theta}}_{N,\alpha} - \boldsymbol{\theta}^*). \tag{23}$$

Now, $\sqrt{n}(\hat{\boldsymbol{\theta}}_{N,\alpha} - \boldsymbol{\theta}^*) \xrightarrow[n\to\infty]{L} N_p(\mathbf{0}, \boldsymbol{\Sigma}_\alpha(\boldsymbol{\theta}^*))$ and using the properties of the normal distribution we get the desired result.

Based on the previous result we can give an approximation to the power function for the testing procedure based on the Wald-type statistics defined in equation (18).

Remark 1. *The power function of the testing procedure based on the Wald-type test statistics, given in Equation (18), in* $\boldsymbol{\theta}^*$ *is given by:*

$$\begin{aligned}
\pi^n_{N,\alpha}(\boldsymbol{\theta}^*) &= P_{\boldsymbol{\theta}=\boldsymbol{\theta}^*}\left(W_n^C(\boldsymbol{\theta}_0) > \chi^2_{p,\beta}\right)\\
&= P_{\boldsymbol{\theta}=\boldsymbol{\theta}^*}\left(n\ l(\hat{\boldsymbol{\theta}}_{N,\alpha}) > \chi^2_{p,\beta}\right)\\
&= P_{\boldsymbol{\theta}=\boldsymbol{\theta}^*}\left(n\ l(\hat{\boldsymbol{\theta}}_{N,\alpha}) - n\ l(\boldsymbol{\theta}^*) > \chi^2_{p,\beta} - n\ l(\boldsymbol{\theta}^*)\right)\\
&= P_{\boldsymbol{\theta}=\boldsymbol{\theta}^*}\left(\sqrt{n}\ (l(\hat{\boldsymbol{\theta}}_{N,\alpha}) - l(\boldsymbol{\theta}^*)) > \frac{\chi^2_{p,\beta}}{\sqrt{n}} - \sqrt{n}\ l(\boldsymbol{\theta}^*)\right)\\
&= P_{\boldsymbol{\theta}=\boldsymbol{\theta}^*}\left(\sqrt{n}\ \frac{(l(\hat{\boldsymbol{\theta}}_{N,\alpha}) - l(\boldsymbol{\theta}^*))}{\sigma_W(\boldsymbol{\theta}^*)} > \frac{\frac{\chi^2_{p,\beta}}{\sqrt{n}} - \sqrt{n}\ l(\boldsymbol{\theta}^*)}{\sigma_W(\boldsymbol{\theta}^*)}\right)\\
&= 1 - \Phi_n\left(\frac{\frac{\chi^2_{p,\beta}}{\sqrt{n}} - \sqrt{n}\ l(\boldsymbol{\theta}^*)}{\sigma_W(\boldsymbol{\theta}^*)}\right)
\end{aligned}$$

where $\Phi_n(x)$ *tends uniformly to the standard normal distribution function* $\Phi(x)$ *and* $\sigma^2_W(\boldsymbol{\theta}^*)$ *is given in Theorem 2. Based on this result, an approximation of the power function of the Wald-type test statistics at* $\boldsymbol{\theta}^*$ *is* $1 - \Phi\left(\frac{\frac{\chi^2_{p,\beta}}{\sqrt{n}} - \sqrt{n}\ l(\boldsymbol{\theta}^*)}{\sigma_W(\boldsymbol{\theta}^*)}\right)$.

Now we are going to derive the asymptotic distribution of Wald-type test statistics, $W_n^C(\boldsymbol{\theta}_0)$, under contiguous alternative hypotheses described by:

$$H_{1,n} : \boldsymbol{\theta}_n = \boldsymbol{\theta}_0 + \frac{1}{\sqrt{n}}\boldsymbol{d} \tag{24}$$

where $\boldsymbol{d}$ is a fixed vector in $\mathbb{R}^p$ such that $\boldsymbol{\theta}_n \in \Theta \subseteq \mathbb{R}^p$ for all n.

Theorem 3. *Under the contiguous alternative hypotheses given in Equation (24), the asymptotic distribution of the proposed Wald-type test statistics* $W_n^C(\boldsymbol{\theta}_0)$ *is a non-central* χ^2 *with* p *degrees of freedom and non-centrality parameter*

$$\delta^C = \boldsymbol{d}^T \boldsymbol{\Sigma}_\alpha(\boldsymbol{\theta}_0)\boldsymbol{d}. \tag{25}$$

Proof:

$$\sqrt{n}(\hat{\boldsymbol{\theta}}_{N,\alpha} - \boldsymbol{\theta}_0) = \sqrt{n}(\hat{\boldsymbol{\theta}}_{N,\alpha} - \boldsymbol{\theta}_n) + \sqrt{n}(\boldsymbol{\theta}_n - \boldsymbol{\theta}_0) = \sqrt{n}(\hat{\boldsymbol{\theta}}_{N,\alpha} - \boldsymbol{\theta}_n) + \boldsymbol{d}$$

Under $H_{1,n}$ *it follows that,*

$$\sqrt{n}(\hat{\boldsymbol{\theta}}_{N,\alpha} - \boldsymbol{\theta}_n) \xrightarrow[n\to\infty]{L} N_p(\mathbf{0}, \boldsymbol{\Sigma}_\alpha(\boldsymbol{\theta}_0))$$

and

$$\sqrt{n}(\hat{\boldsymbol{\theta}}_{N,\alpha} - \boldsymbol{\theta}_0) \xrightarrow[n\to\infty]{L} N_p(\boldsymbol{d}, \boldsymbol{\Sigma}_\alpha(\boldsymbol{\theta}_0)).$$

On the other hand, $W_n^C = \boldsymbol{Y}^T\boldsymbol{Y}$*, where* $\boldsymbol{Y} = \boldsymbol{\Sigma}_\alpha^{-\frac{1}{2}}(\boldsymbol{\theta}_0)(\hat{\boldsymbol{\theta}}_{N,\alpha} - \boldsymbol{\theta}_0)$*, and under* $H_{1,n}$*,*

$$\boldsymbol{Y} \xrightarrow[n\to\infty]{L} N_p(\boldsymbol{\Sigma}_\alpha^{-\frac{1}{2}}(\boldsymbol{\theta}_0)\boldsymbol{d}, \boldsymbol{I}_p)$$

where $\boldsymbol{I}_p$ *is the identity matrix of order* p*. Therefore using the properties for obtaining the asymptotic distribution of quadratic forms, we have* $W_n^C(\boldsymbol{\theta}_0) = \boldsymbol{Y}^T\boldsymbol{Y} \xrightarrow[n\to\infty]{L} \chi_p^2(\delta^C)$ *with* $\delta^C = \boldsymbol{d}^T\boldsymbol{\Sigma}_\alpha(\boldsymbol{\theta}_0)\boldsymbol{d}$*; here* $\chi_p^2(\delta^C)$ *denotes a non-central* χ^2 *distribution with* p *degrees of freedom and non-centrality parameter* δ^C*.*

The result of this theorem can be used to give an approximation of the power function at $\boldsymbol{\theta}^*$. We have $\boldsymbol{\theta}^* = \boldsymbol{\theta}_0 + (\boldsymbol{\theta}^* - \boldsymbol{\theta}_0) = \boldsymbol{\theta}_0 + \frac{1}{\sqrt{n}}\sqrt{n}(\boldsymbol{\theta}^* - \boldsymbol{\theta}_0)$. Now considering $\boldsymbol{d} = \sqrt{n}(\boldsymbol{\theta}^* - \boldsymbol{\theta}_0)$, we can apply the previous theorem.

3.2 *The Composite Null Hypothesis*

We will consider a composite null hypothesis with the restricted parameter space $\Theta_0 \subseteq \Theta$ defined through a set of r restrictions of the form:

$$\boldsymbol{m}(\boldsymbol{\theta}) = \boldsymbol{0} \tag{26}$$

where $\boldsymbol{m} : \mathbb{R}^p \to \mathbb{R}^r$. Assume that the $p \times r$ matrix

$$\boldsymbol{M}(\boldsymbol{\theta}) = \frac{\partial \boldsymbol{m}^T(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \tag{27}$$

exists and is continuous in $\boldsymbol{\theta}$, and $rank(\boldsymbol{M}(\boldsymbol{\theta})) = r$ where $r \leq p$. We will test the hypothesis

$$H_0 : \boldsymbol{\theta} \in \Theta_0 \quad \text{against} \quad H_1 : \boldsymbol{\theta} \notin \Theta_0. \tag{28}$$

Definition 2. *Let* $\hat{\boldsymbol{\theta}}_{N,\alpha}$ *be the MCDE of* $\boldsymbol{\theta}$ *and the family of Wald-type test statistics for testing the hypothesis given in Equation (28) is defined as:*

$$^*W_n^C(\hat{\boldsymbol{\theta}}_{N,\alpha}) = n\ \boldsymbol{m}^T(\hat{\boldsymbol{\theta}}_{N,\alpha})[\boldsymbol{M}^T(\hat{\boldsymbol{\theta}}_{N,\alpha})\boldsymbol{\Sigma}_\alpha(\hat{\boldsymbol{\theta}}_{N,\alpha})\boldsymbol{M}(\hat{\boldsymbol{\theta}}_{N,\alpha})]^{-1}\boldsymbol{m}(\hat{\boldsymbol{\theta}}_{N,\alpha}) \tag{29}$$

where $\boldsymbol{\Sigma}_\alpha(\boldsymbol{\theta})$*,* $\boldsymbol{m}$ *and* $\boldsymbol{M}$ *are defined in Equations (10), (26) and (27) respectively.*

Theorem 4. *The asymptotic null distribution of the proposed Wald-type test statistics given in Equation (29) is* χ^2 *with* r *degrees of freedom.*

Proof: *Let* $\boldsymbol{\theta}_0 \in \Theta_0$ *be the true value of* $\boldsymbol{\theta}$*. Using a Taylor series expansion of* $\boldsymbol{m}(\boldsymbol{\theta})$ *at* $\hat{\boldsymbol{\theta}}_{N,\alpha}$ *around* $\boldsymbol{\theta}_0$ *we get*

$$\begin{aligned} \boldsymbol{m}(\hat{\boldsymbol{\theta}}_{N,\alpha}) &= \boldsymbol{m}(\boldsymbol{\theta}_0) + \boldsymbol{M}^T(\boldsymbol{\theta}_0)(\hat{\boldsymbol{\theta}}_{N,\alpha} - \boldsymbol{\theta}_0) + o_p(||\hat{\boldsymbol{\theta}}_{N,\alpha} - \boldsymbol{\theta}_0||) \\ &= \boldsymbol{M}^T(\boldsymbol{\theta}_0)(\hat{\boldsymbol{\theta}}_{N,\alpha} - \boldsymbol{\theta}_0) + o_p(||\hat{\boldsymbol{\theta}}_{N,\alpha} - \boldsymbol{\theta}_0||) \end{aligned} \tag{30}$$

since from Equation (26), we have $\boldsymbol{m}(\boldsymbol{\theta}) = \boldsymbol{0}$*. Now under* H_0*,* $\sqrt{n}(\hat{\boldsymbol{\theta}}_{N,\alpha} - \boldsymbol{\theta}_0) \xrightarrow[n\to\infty]{L}$

$N_p(\mathbf{0}, \boldsymbol{\Sigma}_\alpha(\boldsymbol{\theta}_0))$. Therefore, from Equation (30) we get, under H_0,

$$\sqrt{n}\ \boldsymbol{m}(\hat{\boldsymbol{\theta}}_{N,\alpha}) \xrightarrow[n\to\infty]{L} N_p(\mathbf{0}, \boldsymbol{M}^T(\boldsymbol{\theta}_0)\boldsymbol{\Sigma}_\alpha(\boldsymbol{\theta}_0)\boldsymbol{M}(\boldsymbol{\theta}_0))$$

As $rank(\boldsymbol{M}(\boldsymbol{\theta})) = r$, we get

$$n\ \boldsymbol{m}^T(\hat{\boldsymbol{\theta}}_{N,\alpha})[\boldsymbol{M}^T(\boldsymbol{\theta}_0)\boldsymbol{\Sigma}_\alpha(\boldsymbol{\theta}_0)\boldsymbol{M}(\boldsymbol{\theta}_0)]^{-1}\boldsymbol{m}(\hat{\boldsymbol{\theta}}_{N,\alpha}) \xrightarrow[n\to\infty]{L} \chi^2_r$$

Now since $\boldsymbol{m}^T(\hat{\boldsymbol{\theta}}_{N,\alpha})[\boldsymbol{M}^T(\hat{\boldsymbol{\theta}}_{N,\alpha})\boldsymbol{\Sigma}_\alpha(\hat{\boldsymbol{\theta}}_{N,\alpha})\boldsymbol{M}(\hat{\boldsymbol{\theta}}_{N,\alpha})]^{-1}\boldsymbol{m}(\hat{\boldsymbol{\theta}}_{N,\alpha})$ is a consistent estimator of

$$\boldsymbol{m}^T(\hat{\boldsymbol{\theta}}_{N,\alpha})[\boldsymbol{M}^T(\boldsymbol{\theta}_0)\boldsymbol{\Sigma}_\alpha(\boldsymbol{\theta}_0)\boldsymbol{M}(\boldsymbol{\theta}_0)]^{-1}\boldsymbol{m}(\hat{\boldsymbol{\theta}}_{N,\alpha})$$

hence under H_0,

$$n\ \boldsymbol{m}^T(\hat{\boldsymbol{\theta}}_{N,\alpha})[\boldsymbol{M}^T(\hat{\boldsymbol{\theta}}_{N,\alpha})\boldsymbol{\Sigma}_\alpha(\hat{\boldsymbol{\theta}}_{N,\alpha})\boldsymbol{M}(\hat{\boldsymbol{\theta}}_{N,\alpha})]^{-1}\boldsymbol{m}(\hat{\boldsymbol{\theta}}_{N,\alpha}) \xrightarrow[n\to\infty]{L} \chi^2_r$$

Therefore, we reject the null hypothesis given in Equation (28) if

$$^{*}W_n^C(\hat{\boldsymbol{\theta}}_{N,\alpha}) > \chi^2_{p,\beta} \tag{31}$$

The following theorem may be used to approximate the power function for the Wald-type test statistics given in Equation (31). Assume that $\boldsymbol{\theta}^* \notin \Theta_0(\in \Theta)$ is the true value of the parameter so that the unrestricted estimator $\hat{\boldsymbol{\theta}}_{N,\alpha} \xrightarrow[n\to\infty]{P} \boldsymbol{\theta}^*$.

Theorem 5. *Let $l^*(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) = \boldsymbol{m}^T(\boldsymbol{\theta}_1)\left[\boldsymbol{M}^T(\boldsymbol{\theta}_2)\boldsymbol{\Sigma}_\alpha(\boldsymbol{\theta}_2)\boldsymbol{M}(\boldsymbol{\theta}_2)\right]^{-1}\boldsymbol{m}(\boldsymbol{\theta}_1)$. Then*

$$\sqrt{n}\left(l^*\left(\hat{\boldsymbol{\theta}}_{N,\alpha}, \hat{\boldsymbol{\theta}}_{N,\alpha}\right) - l^*\left(\boldsymbol{\theta}^*, \boldsymbol{\theta}^*\right)\right) \xrightarrow[n\to\infty]{L} N_p(\mathbf{0}, \sigma^{*2}_W(\boldsymbol{\theta}^*)), \tag{32}$$

where

$$\sigma^{*2}_W(\boldsymbol{\theta}^*) = \left(\frac{\partial l^*(\boldsymbol{\theta}, \boldsymbol{\theta}^*)}{\partial \boldsymbol{\theta}}\right)^T_{\boldsymbol{\theta}=\boldsymbol{\theta}^*} \boldsymbol{\Sigma}_\alpha(\boldsymbol{\theta}^*) \left(\frac{\partial l^*(\boldsymbol{\theta}, \boldsymbol{\theta}^*)}{\partial \boldsymbol{\theta}}\right)_{\boldsymbol{\theta}=\boldsymbol{\theta}^*}. \tag{33}$$

***Proof**: We note that $\hat{\boldsymbol{\theta}}_{N,\alpha} \xrightarrow[n\to\infty]{P} \boldsymbol{\theta}^*$ and using this result we get that $l^*\left(\hat{\boldsymbol{\theta}}_{N,\alpha}, \hat{\boldsymbol{\theta}}_{N,\alpha}\right)$ and $l^*\left(\hat{\boldsymbol{\theta}}_{N,\alpha}, \boldsymbol{\theta}^*\right)$ have the same asymptotic distribution. Now a Taylor series expansion of $l^*(\hat{\boldsymbol{\theta}}_{N,\alpha}, \boldsymbol{\theta}^*)$ around $\boldsymbol{\theta}^*$ gives*

$$l^*\left(\hat{\boldsymbol{\theta}}_{N,\alpha}, \hat{\boldsymbol{\theta}}_{N,\alpha}\right) - l^*\left(\boldsymbol{\theta}^*, \boldsymbol{\theta}^*\right) = \left(\frac{\partial l^*(\boldsymbol{\theta}, \boldsymbol{\theta}^*)}{\partial \boldsymbol{\theta}}\right)_{\boldsymbol{\theta}=\boldsymbol{\theta}^*} (\hat{\boldsymbol{\theta}}_{N,\alpha} - \boldsymbol{\theta}^*) + o_p(||\hat{\boldsymbol{\theta}}_{N,\alpha} - \boldsymbol{\theta}^*||). \tag{34}$$

Using the properties of the normal distribution on Equation (9) we get the desired result.

Remark 2. *The power function of the Wald-type test statistics defined in Equation (31)* *at $\boldsymbol{\theta}^*$ is given by:*

$$\begin{aligned}
\pi_{N,\alpha}^{*,n} &= P_{\boldsymbol{\theta}=\boldsymbol{\theta}^*}\left({}^*W_n^C(\hat{\boldsymbol{\theta}}_{N,\alpha}) > \chi^2_{r,\beta}\right)\\
&= P_{\boldsymbol{\theta}=\boldsymbol{\theta}^*}\left(n\ l^*\left(\hat{\boldsymbol{\theta}}_{N,\alpha},\hat{\boldsymbol{\theta}}_{N,\alpha}\right) > \chi^2_{r,\beta}\right)\\
&= P_{\boldsymbol{\theta}=\boldsymbol{\theta}^*}\left(n\ l^*\left(\hat{\boldsymbol{\theta}}_{N,\alpha},\hat{\boldsymbol{\theta}}_{N,\alpha}\right) - n\ l^*(\boldsymbol{\theta}^*,\boldsymbol{\theta}^*) > \chi^2_{r,\beta} - n\ l^*(\boldsymbol{\theta}^*,\boldsymbol{\theta}^*)\right)\\
&= P_{\boldsymbol{\theta}=\boldsymbol{\theta}^*}\left(\frac{n\ l^*\left(\hat{\boldsymbol{\theta}}_{N,\alpha},\hat{\boldsymbol{\theta}}_{N,\alpha}\right) - n\ l^*(\boldsymbol{\theta}^*,\boldsymbol{\theta}^*)}{\sigma^*_W(\boldsymbol{\theta}^*)} > \frac{\chi^2_{r,\beta} - n\ l^*(\boldsymbol{\theta}^*,\boldsymbol{\theta}^*)}{\sigma^*_W(\boldsymbol{\theta}^*)}\right)\\
&= 1-\Phi_n\left(\frac{\chi^2_{r,\beta} - n\ l^*(\boldsymbol{\theta}^*,\boldsymbol{\theta}^*)}{\sigma^*_W(\boldsymbol{\theta}^*)}\right)
\end{aligned}$$

where $\Phi_n(x)$ tends uniformly to the standard normal distribution function $\Phi(x)$, $\chi^2_{r,\beta}$ be the $(1-\beta)$-th quantile of χ^2 distribution with r degrees of freedom and $\sigma^*_W(\boldsymbol{\theta}^*)$ is given in Theorem 5.

The power function of ${}^*W_n^C(\hat{\boldsymbol{\theta}}_{N,\alpha})$ at an alternative close to the null hypothesis may be approximated using contiguous alternative hypothesis. Let $\boldsymbol{\theta}_n \in \Theta - \Theta_0$ be a given alternative, and let $\boldsymbol{\theta}_0$ be the element in Θ_0 closest to $\boldsymbol{\theta}_n$ in terms of the Euclidean distance. We may introduce contiguous alternative hypotheses by considering a fixed $\boldsymbol{d} \in \mathbb{R}^p$ and by permitting $\boldsymbol{\theta}_n$ to move towards $\boldsymbol{\theta}_0$ as n increases through the relation:

$$H_{1,n} : \boldsymbol{\theta}_n = \boldsymbol{\theta}_0 + \frac{1}{\sqrt{n}}\boldsymbol{d}. \tag{35}$$

Another approach will be by relaxing the condition $\boldsymbol{m}(\boldsymbol{\theta}) = \boldsymbol{0_r}$ while defining Θ_0. Consider the following sequence of parameters $\{\boldsymbol{\theta}_n\}$ moving towards $\boldsymbol{\theta}_0$ according to the set up:

$$H^*_{1,n} : \boldsymbol{m}(\boldsymbol{\theta}_n) = \frac{1}{\sqrt{n}}\boldsymbol{\delta}, \tag{36}$$

where $\boldsymbol{\delta} \in \mathbb{R}^r$. Now a first order Taylor series expansion of $\boldsymbol{m}(\boldsymbol{\theta}_n)$ around $\boldsymbol{\theta}_0$ gives:

$$\boldsymbol{m}(\boldsymbol{\theta}_n) = \boldsymbol{m}(\boldsymbol{\theta}_0) + \boldsymbol{M}^T(\boldsymbol{\theta}_0)(\boldsymbol{\theta}_n - \boldsymbol{\theta}_0) + o(||\boldsymbol{\theta}_n - \boldsymbol{\theta}_0||). \tag{37}$$

Using the relation $\boldsymbol{\theta}_n = \boldsymbol{\theta}_0 + \frac{1}{\sqrt{n}}\boldsymbol{d}$ and $\boldsymbol{m}(\boldsymbol{\theta}_0) = 0$ in Equation (37), we get:

$$\boldsymbol{m}(\boldsymbol{\theta}_n) = \frac{1}{\sqrt{n}}\boldsymbol{M}^T(\boldsymbol{\theta}_0)\boldsymbol{d} + o(||\boldsymbol{\theta}_n - \boldsymbol{\theta}_0||) \tag{38}$$

The equivalence relationship between $H_{1,n}$ and $H^*_{1,n}$ is $\boldsymbol{\delta} = \boldsymbol{M}^T(\boldsymbol{\theta}_0)\boldsymbol{d}$ when $n \to \infty$. In the following theorem, we show the asymptotic distributions of the Wald-type test statistics ${}^*W_n^C(\hat{\boldsymbol{\theta}}_{N,\alpha})$ under the alternative hypotheses $H_{1,n}$ and $H^*_{1,n}$ as given by Equations (35) and (36), respectively.

Theorem 6. *The asymptotic distribution of ${}^*W_n^C(\hat{\boldsymbol{\theta}}_{N,\alpha})$ is given by*

1. *Under* $H_{1,n}$, ${}^*W_n^C(\hat{\boldsymbol{\theta}}_{N,\alpha}) \xrightarrow[n\to\infty]{L} \chi_r^2(a)$, *$a$ being the parameter of non-centrality is given by:*
$$a = \left(\boldsymbol{d}^T\boldsymbol{M}(\boldsymbol{\theta}_0)\left[\boldsymbol{M}^T(\boldsymbol{\theta}_0)\boldsymbol{\Sigma}_\alpha(\boldsymbol{\theta}_0)\boldsymbol{M}(\boldsymbol{\theta}_0)\right]^{-1}\boldsymbol{M}^T(\boldsymbol{\theta}_0)\boldsymbol{d}\right)$$

2. *Under* $H_{1,n}^*$, ${}^*W_n^C(\hat{\boldsymbol{\theta}}_{N,\alpha}) \xrightarrow[n\to\infty]{L} \chi_r^2(b)$, *$b$ being the parameter of non-centrality is given by:*
$$b = \left(\boldsymbol{\delta}^T\left[\boldsymbol{M}^T(\boldsymbol{\theta}_0)\boldsymbol{\Sigma}_\alpha(\boldsymbol{\theta}_0)\boldsymbol{M}(\boldsymbol{\theta}_0)\right]^{-1}\boldsymbol{\delta}\right)$$

Proof: *A first order Taylor series expansion of* $\boldsymbol{m}(\hat{\boldsymbol{\theta}}_{N,\alpha})$ *around* $\boldsymbol{\theta}_n$ *gives:*

$$\boldsymbol{m}(\hat{\boldsymbol{\theta}}_{N,\alpha}) = \boldsymbol{m}(\boldsymbol{\theta}_n) + \boldsymbol{M}^T(\boldsymbol{\theta}_n)(\hat{\boldsymbol{\theta}}_{N,\alpha} - \boldsymbol{\theta}_n) + o(||\hat{\boldsymbol{\theta}}_{N,\alpha} - \boldsymbol{\theta}_n||)$$

Using Equation (38) we have:

$$\boldsymbol{m}(\hat{\boldsymbol{\theta}}_{N,\alpha}) = \frac{1}{\sqrt{n}}\boldsymbol{M}^T(\boldsymbol{\theta}_0)\boldsymbol{d} + \boldsymbol{M}^T(\boldsymbol{\theta}_n)(\hat{\boldsymbol{\theta}}_{N,\alpha} - \boldsymbol{\theta}_n) + o(||\hat{\boldsymbol{\theta}}_{N,\alpha} - \boldsymbol{\theta}_n||) + o(||\boldsymbol{\theta}_n - \boldsymbol{\theta}_0||) \quad (39)$$

Under $H_{1,n}$ *we get* $\sqrt{n}(\hat{\boldsymbol{\theta}}_{N,\alpha} - \boldsymbol{\theta}_n) \xrightarrow[n\to\infty]{L} N_r(\boldsymbol{0}, \boldsymbol{\Sigma}_\alpha(\boldsymbol{\theta}_0))$ *Thus from Equation (39) we have:*

$$\sqrt{n}\ \boldsymbol{m}(\hat{\boldsymbol{\theta}}_{N,\alpha}) \xrightarrow[n\to\infty]{L} N_r(\boldsymbol{M}^T(\boldsymbol{\theta}_0)\boldsymbol{d}, \boldsymbol{M}^T(\boldsymbol{\theta}_0)\boldsymbol{\Sigma}_\alpha(\boldsymbol{\theta}_0)\boldsymbol{M}(\boldsymbol{\theta}_0))$$

From Equation (36) we get, under $H_{1,n}^*$, $\sqrt{n}\ \boldsymbol{m}(\hat{\boldsymbol{\theta}}_{N,\alpha}) \xrightarrow[n\to\infty]{L} N_r(\boldsymbol{\delta}, \boldsymbol{M}^T(\boldsymbol{\theta}_0)\boldsymbol{\Sigma}_\alpha(\boldsymbol{\theta}_0)\boldsymbol{M}(\boldsymbol{\theta}_0))$. *We may write* ${}^*W_n^C(\hat{\boldsymbol{\theta}}_{N,\alpha})$ *as* $\boldsymbol{Z}^T\boldsymbol{Z}$ *where:*

$$\boldsymbol{Z} = \sqrt{n}\ \boldsymbol{m}(\hat{\boldsymbol{\theta}}_{N,\alpha})[\boldsymbol{M}^T(\boldsymbol{\theta}_0)\boldsymbol{\Sigma}_\alpha(\boldsymbol{\theta}_0)\boldsymbol{M}(\boldsymbol{\theta}_0)]^{-\frac{1}{2}}$$

We know $\boldsymbol{Z} \xrightarrow[n\to\infty]{L} N_r([\boldsymbol{M}^T(\boldsymbol{\theta}_0)\boldsymbol{\Sigma}_\alpha(\boldsymbol{\theta}_0)\boldsymbol{M}(\boldsymbol{\theta}_0)]^{-\frac{1}{2}}\boldsymbol{M}^T(\boldsymbol{\theta}_0)\boldsymbol{d}, \boldsymbol{I}_r)$, *where* $\boldsymbol{I}_r$ *is the identity matrix of order* r. *Using a standard result regarding the quadratic form of a normal variable we get the desired result with the non-centrality parameter* $\boldsymbol{d}^T\boldsymbol{M}(\boldsymbol{\theta}_0)[\boldsymbol{M}^T(\boldsymbol{\theta}_0)\boldsymbol{\Sigma}_\alpha(\boldsymbol{\theta}_0)\boldsymbol{M}(\boldsymbol{\theta}_0)]^{-1}\boldsymbol{M}^T(\boldsymbol{\theta}_0)\boldsymbol{d}$.

4. Wald-type Tests Based on the MCDE: Robust Results

We will now study the robustness of the proposed Wald-type tests discussed in Section 3 with the help of the influence function of the corresponding test statistics W_n^C and ${}^*W_n^C$ defined in Definitions 1 and 2, respectively. Ignoring the multiplier n, let us define the associated statistical functional for the Wald-type test statistics W_n^C evaluated at any distribution G which is given by

$$W_n^C(G) = (\boldsymbol{T}(G) - \boldsymbol{\theta}_0)^T\boldsymbol{\Sigma}_\alpha^{-1}(\boldsymbol{\theta}_0)(\boldsymbol{T}(G) - \boldsymbol{\theta}_0) \quad (40)$$

Consider the contaminated distribution $G_\epsilon = (1-\epsilon)G + \epsilon\omega_v$ where ω_v is the point mass distribution at the point v. The influence function of $W_n^C(G)$ is defined as:

$$\begin{aligned} IF(v, W_n^C, G) &= \left.\frac{\partial W_n^C(G_\epsilon)}{\partial \epsilon}\right|_{\epsilon=0} \\ &= (\boldsymbol{T}(G) - \boldsymbol{\theta}_0)^T \boldsymbol{\Sigma}_\alpha^{-1}(\boldsymbol{\theta}_0) IF(v, \boldsymbol{T}, G) \end{aligned} \quad (41)$$

where $IF(v, \boldsymbol{T}, G)$ is the first order influence function of the minimum C-divergence estimator (for details see Maji *et al.*, 2018). Assuming the null hypothesis to be true and under the true model we have $\boldsymbol{T}(F_{\boldsymbol{\theta}_0}) = \boldsymbol{\theta}_0$ and hence:

$$IF(v, W_n^C, F_{\boldsymbol{\theta}_0}) = 0$$

Thus, we need to go for the higher order influence function of the test statistics to evaluate the robustness property of the corresponding test. The second order influence function of the test statistics W_n^C at the null model distribution $F_{\boldsymbol{\theta}_0}$ is given by

$$IF_2(v, W_n^C, F_{\boldsymbol{\theta}_0}) = \left.\frac{\partial^2 W_n^C(G_\epsilon)}{\partial \epsilon^2}\right|_{\epsilon=0}.$$

By further calculations we get:

$$IF_2(v, W_n^C, F_{\boldsymbol{\theta}_0}) = IF(v, \boldsymbol{T}, F_{\boldsymbol{\theta}_0})^T \boldsymbol{\Sigma}_\alpha^{-1}(\boldsymbol{\theta}_0) IF(v, \boldsymbol{T}, F_{\boldsymbol{\theta}_0}) \quad (42)$$

where $IF(v, \boldsymbol{T}, F_{\boldsymbol{\theta}_0})$ has the form:

$$IF(v, \boldsymbol{T}, F_{\boldsymbol{\theta}_0}) = \left[\int f_{\boldsymbol{\theta}_0}^{1+\alpha}(x) u_{\boldsymbol{\theta}_0}(x) u_{\boldsymbol{\theta}_0}^T(x)\ dx\right]^{-1} \left\{u_{\boldsymbol{\theta}_0}(v) f_{\boldsymbol{\theta}_0}^{\alpha}(v) - \int u_{\boldsymbol{\theta}_0}(x) f_{\boldsymbol{\theta}_0}^{1+\alpha}(x)\ dx\right\}. \quad (43)$$

The proof is straightforward and thus it is omitted. Now we will find the influence function of the Wald-type test statistics ${}^*W_n^C$ in a similar way. Following the same argument as for ${}^*W_n^C$, it is easy to show that the first order influence function is zero for this Wald-type test statistics also. The second order influence function of ${}^*W_n^C$ under the true model has the form:

$$IF_2(v, {}^*W_n^C, F_{\boldsymbol{\theta}_0}) = IF(v, \boldsymbol{T}, F_{\boldsymbol{\theta}_0})^T \boldsymbol{M}(\boldsymbol{\theta}_0) \boldsymbol{\Sigma}_\alpha^{-1}(\boldsymbol{\theta}_0) \boldsymbol{M}^T(\boldsymbol{\theta}_0) IF(v, \boldsymbol{T}, F_{\boldsymbol{\theta}_0}) \quad (44)$$

5. Simulation Study

In this section, we show simulation analysis using the $\text{GPD}_{\alpha,\lambda}$ family, defined in Equation (3). Maji *et al.* (2018) have shown that both the CR family (for $\alpha = 0$) and the DPD family (scaled version for $\alpha = \lambda$) may be found as particular members of the $\text{GPD}_{\alpha,\lambda}$ family. We will show that there exist several members of the $\text{GPD}_{\alpha,\lambda}$ family which show good robustness properties but do not belong to either the CR family or the

DPD family. Though we have theoretically proved the distributional result, those results are asymptotic in nature. In this section, we show that we may get the desired results for small sample size also.

5.1 *The case of the simple null hypothesis*

We will start the section with the following simulation study. We have taken the Poisson model to calculate the level and the power whereas the hypothesis is taken to be $H_0 : \theta = 5$ against $H_1 : \theta \neq 5$. The sample size has been kept as $n = 20$ and the exercise has been repeated 1000 times whereas the nominal level is 5 per cent. We have taken data from the Poisson model with parameter $\theta = 5$ to calculate the level under the pure data and from the 90 percent Poisson$(5) + 10$ percent Poisson(25) model to calculate the level under the contaminated data. The simulated levels under the pure model are presented in Table 1, whereas the simulated levels under the contaminated data are presented in Table 2. For pure data, the tests corresponding to small values of α and small positive values of λ produce tests which have the closest match between the nominal and observed levels. Table 2 shows good robust results for $\alpha = 0.2, 0.3$ and $-0.2 \leq \lambda \leq 0.2$. Similarly, we get robust results for other set of values like $\alpha = 0.4, 0.5, 0.6$ and $-0.1 \leq \lambda \leq 0.4$. The observed levels in this region are least effected by the contamination. The robust region, marked bold in Table 2, moves to more positive values of λ for higher values of α. Through this study, we can easily show that there are several set of values under which the test gives robust results but does not fall under the region of either the CR or the DPD family. We have taken data from the Poisson model with parameter $\theta = 3$ to calculate the power under the pure data and from the 90 percent Poisson$(3) + 10$ percent Poisson(15) model to calculate the power under the contaminated data. The powers under the pure model and the contaminated model are presented in Tables 3 and 4 respectively. For pure data, the power is always close to 1. For contaminated data, the power values do not get distorted for negative λ even for smaller values of α whereas as we move towards higher values of α, we get good powers for positive λ also. Like Table 2, Table 4 shows good simulated powers for which the divergences do not belong to either the CR or the DPD family.

5.2 *The case of the composite null hypothesis*

To explore the performance of our proposed Wald-type test statistics in case of the composite null hypothesis, we have performed a simulation study for the case of a normal population $N(\mu, \sigma^2)$. We consider the hypothesis $H_0 : \mu = 0$ against the alternative $H_1 : \mu \neq 0$ with σ unknown. In this case, the parameter space is given by $\Theta = \{(\mu, \sigma) \in \mathbb{R}^2 \mid \mu \in \mathbb{R}, \sigma \in \mathbb{R}^+\}$, and the parameter space under the null distribution is $\Theta_0 = \{(\mu, \sigma) \in \mathbb{R}^2 \mid \mu = 0, \sigma \in \mathbb{R}^+\}$. If we consider the function $\boldsymbol{m}(\boldsymbol{\theta}) = \mu$ where $\boldsymbol{\theta} = (\mu, \sigma)^T$, the null hypothesis H_0 can be written as $H_0 : \boldsymbol{m}(\boldsymbol{\theta}) = 0$. We observe that in our case $\boldsymbol{M}(\boldsymbol{\theta}) = (1, 0)^T$. The Wald-type test statistics used in this example are given by Equation (29). We have used the smoothed model approach (Basu and

Lindsay, 1994) where the model is smoothed with the same kernel function as with the data. The idea here is to find a minimized distance between the data and the model. The procedure usually make the data continuous by introducing a kernel and at the same time, it should be kept in mind that the effect of this kernel to be made as minimal as possible. It is for this reason both the model and the data are convoluted by the same kernel so that the distortion made by the kernel to the data is nullified by the use of the same kernel to the model. In order to do the simulation study we generate data from the $N(0, 1)$ distribution. Subsequently, the same hypotheses were tested when the data were generated from the $N(-1, 1)$ distribution. Table 5 gives the simulated level whereas Table 6 gives the simulated power under the pure normal model. For pure data, the tests seem to be conservative for most values of α and λ as the calculated levels are less than 0.05 for most of the cases though the power values are high for most cases. Now, we show the performance of the proposed Wald-type tests under contamination. So, we have tested the same hypothesis, but the data have been generated from the $0.9\ N(0, 1)\ +\ 0.1(10, 1)$ distribution to calculate the level under contamination. Finally, we have generated data from the normal mixture $0.9\ N(-1, 1)\ +\ 0.1(10, 1)$ to calculate the power under contamination. Tables 7 and 8 give the simulated level and the simulated power under the contaminated normal model respectively. For contaminated data, the calculated levels are less distorted for negative λ even for low values of α. The calculated levels are close to the desired levels for positive λ for higher values of α. The power values are less distorted for negative λ by the contamination and the tests have less power as we move towards positive λ and for lower values of α.

6. Real Data Example

6.1 *Drosophila data*

We consider the data presented by Woodruff *et al.* (1984) involving a sex linked recessive lethal test in drosophila (fruit flies). The data (Table 9) shows the frequencies of number of recessive lethal mutations observed among the daughters of male flies exposed to certain doses of a chemical. The observations at values 3 and 4 appear to represent moderate outliers. Here we use the Poisson model to estimate the parameter θ and we want to test:

$$H_0 : \theta = 0.11 \quad \text{against} \quad H_1 : \theta \neq 0.11.$$

The choice of the hypothesis comes as a result of the estimation of the parameter after deleting those outliers. For details, see Maji *et al.* (2018), Table 3. We have calculated p-values of the Wald-type tests and presented those values in Tables 10 and 11. Table 10 gives p-values for full data whereas Table 11 gives p-values after deleting those moderate outliers. By analyzing both tables it can be easily seen that for α close to zero and positive λ, the p-values are mostly affected whereas for $\alpha = 0.2, 0.3$ and smaller negative values of λ, the effect of the outliers are seen to be minimal.

6.2 *Normal examples using real data*

In this section, we will analyze two real data sets using a continuous model. The first data set has been previously presented by Welch (1987) and Simpson (1989). The data are from an experiment to test a method of reducing faults on telephone lines. The data in Table 12 pertains 14 matched pairs of areas and provides the ordered differences between the inverse test rates and the inverse control rates in those areas. The first data set shows a large number of outliers. We will use a normal model with parameters μ and σ in this example. Let us now consider testing of the null hypothesis $H_0 : \mu = 0$ against $H_1 : \mu \neq 0$, where σ is an unknown nuisance parameter. The parameter space is given by $\Theta = \{(\mu, \sigma) \in \mathbb{R}^2 | \mu \in \mathbb{R}, \sigma \in \mathbb{R}^+\}$, and the parameter space under the null distribution is $\Theta = \{(\mu, \sigma) \in \mathbb{R}^2 | \mu = \mu_0, \sigma \in \mathbb{R}^+\}$. We may consider the function $\boldsymbol{m}(\boldsymbol{\theta}) = \mu - \mu_0$, where $\boldsymbol{\theta} = (\mu, \sigma)^T$, the null hypothesis H_0 can be written as:

$$H_0 : \boldsymbol{m}(\boldsymbol{\theta}) = 0.$$

In this case $\boldsymbol{M}(\boldsymbol{\theta}) = (1, 0)^T$. To calculate the minimum GPD estimates, we have used the kernel density estimator with the Gaussian kernel for the construction of the divergence. The bandwidth h_n has been taken as $h_n = 1.06\tau_n n^{-1/5}$ where $\tau_n = median_i|X_i - median_j X_j|/0.6745$. The Wald-type test statistics used in this example are given by Equation (29). Table 13 gives p-values for the full data whereas Table 14 gives p-values after deleting the large outlier. The test rejects the null hypothesis for all values of α and λ in case of outliers deleted data whereas the test fails to reject the null hypothesis for positive λ in case of full data.

The second example involves an experiment done by Charles Darwin (Darwin, 1878). Charles Darwin had done an experiment involving various types of plants to show the effect of different fertilization methods. The experiment was used to tell whether self-fertilized plants and cross-fertilized plants have different growth rates. In this experiment, one special type of plant viz. Zea Mays, were planted in pots in pairs, one self-fertilized and the other cross-fertilized and after a time period the height of each plant was measured and a sample of 15 such paired differences (Table 15) between cross-fertilized minus self-fertilized were taken. The hypothesis and the corresponding Wald-type test statistics have been kept same as the example of the telephone fault data. Table 16 gives p-values for the full data whereas Table 17 gives p-values after deleting two negative outliers. The test shows similar results like the telephone fault data.

7. Choice of Tuning Parameter

In this section, we will restrict ourselves to the GPD family and will not go for the general C-divergence as choice of $N(\cdot)$ requires a separate research. Here we will give some idea to the users regarding the values of the tuning parameters α and λ to be used in practical situations. For pure data, we may use $\alpha = \lambda = 0$ as it matches with the likelihood disparity. We may use the data driven approach for contaminated data as we have shown in the above examples. In order to study the overall robustness aspect of the divergence method, we may perform an overall minimization of each of the divergences by considering their tuning parameters to be nuisance parameters varying within a reasonable range. It may be noted that there are several members of the GPD family outside either the CR or the DPD family of divergences which show good robust properties. The region for negative λ and positive α seems to produce good robust results and we usually require high positive value of α for positive λ to get robust results.

8. Conclusion

In this paper, we have proposed a Wald-type test based on the minimum C-divergence estimators. We have shown that there exists a direct distributional form of the Wald-type test statistics and have also developed an asymptotic null distribution of the Wald-type test statistics and the approximation of the power function under simple null hypothesis and composite null hypothesis. We have calculated both first order and second order influence function of the Wald-type test statistics. Both simulated and real data example have shown that there exists a region of the parameters for which the tests are found to be robust. We may extend our approach to other divergences.

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Annexure

Table 1: Simulated levels of the $GPD_{\alpha,\lambda}$ Wald-type tests with pure data under the Poisson model (sample size 20) for various values of λ and α; the nominal level is 5%.

$\lambda \downarrow \alpha \rightarrow$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
−0.9	0.274	0.263	0.252	0.242	0.229	0.223	0.218	0.201	0.198	0.188	0.173
−0.8	0.206	0.199	0.192	0.179	0.176	0.176	0.176	0.166	0.16	0.156	0.149
−0.7	0.143	0.146	0.143	0.145	0.142	0.143	0.149	0.146	0.141	0.139	0.134
−0.6	0.111	0.119	0.12	0.121	0.121	0.123	0.128	0.135	0.133	0.131	0.127
−0.5	0.088	0.104	0.103	0.105	0.111	0.12	0.122	0.125	0.129	0.128	0.129
−0.4	0.074	0.082	0.098	0.101	0.1	0.102	0.109	0.115	0.114	0.118	0.122
−0.3	0.066	0.078	0.079	0.088	0.094	0.096	0.102	0.105	0.108	0.109	0.116
−0.2	0.061	0.067	0.079	0.077	0.086	0.089	0.094	0.1	0.097	0.102	0.107
−0.1	0.055	0.061	0.075	0.076	0.08	0.084	0.084	0.092	0.097	0.097	0.1
0	0.053	0.054	0.069	0.075	0.077	0.084	0.081	0.085	0.095	0.093	0.093
0.1	0.05	0.051	0.057	0.069	0.074	0.075	0.079	0.082	0.087	0.092	0.09
0.2	0.051	0.048	0.052	0.06	0.068	0.069	0.073	0.079	0.081	0.091	0.089
0.3	0.054	0.049	0.053	0.054	0.061	0.068	0.071	0.076	0.079	0.084	0.089
0.4	0.057	0.051	0.05	0.052	0.056	0.063	0.066	0.07	0.076	0.082	0.082
0.5	0.061	0.054	0.049	0.051	0.055	0.057	0.061	0.07	0.07	0.074	0.078
0.6	0.061	0.056	0.05	0.048	0.053	0.057	0.056	0.063	0.066	0.07	0.071
0.7	0.068	0.058	0.053	0.048	0.049	0.055	0.052	0.055	0.063	0.066	0.068
0.8	0.074	0.06	0.052	0.049	0.048	0.053	0.051	0.053	0.057	0.061	0.065
0.9	0.079	0.064	0.058	0.049	0.047	0.05	0.051	0.052	0.054	0.056	0.059
1	0.084	0.073	0.06	0.051	0.048	0.047	0.05	0.047	0.05	0.051	0.056

Table 2: **Simulated levels of the $\mathrm{GPD}_{\alpha,\lambda}$ Wald-type tests with contaminated data under the Poisson model (sample size 20) for various values of λ and α; the nominal level is 5%.**

$\lambda \downarrow \alpha \rightarrow$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
−0.9	0.327	0.317	0.293	0.276	0.263	0.256	0.252	0.237	0.229	0.213	0.205
−0.8	0.243	0.241	0.233	0.226	0.217	0.206	0.196	0.187	0.181	0.174	0.171
−0.7	0.187	0.201	0.194	0.188	0.178	0.174	0.17	0.161	0.159	0.152	0.147
−0.6	0.141	0.156	0.156	0.158	0.153	0.152	0.149	0.145	0.143	0.139	0.138
−0.5	0.12	0.127	0.129	0.13	0.135	0.132	0.129	0.127	0.125	0.12	0.118
−0.4	0.095	0.113	0.115	0.121	0.115	0.119	0.12	0.119	0.113	0.113	0.112
−0.3	0.079	0.101	0.104	0.108	0.107	0.108	0.109	0.11	0.109	0.109	0.112
−0.2	0.086	0.102	**0.094**	**0.099**	0.101	0.103	0.104	0.105	0.105	0.104	0.105
−0.1	0.198	0.107	**0.086**	**0.093**	**0.093**	**0.096**	**0.097**	0.101	0.104	0.1	0.101
0	0.698	0.134	**0.086**	**0.08**	**0.081**	**0.089**	**0.093**	**0.094**	**0.099**	**0.096**	**0.098**
0.1	0.855	0.191	**0.097**	**0.079**	**0.075**	**0.082**	**0.092**	**0.09**	**0.095**	**0.094**	**0.095**
0.2	0.867	0.809	**0.077**	**0.079**	**0.07**	**0.075**	**0.086**	**0.088**	**0.093**	**0.09**	**0.092**
0.3	0.87	0.858	0.739	**0.069**	**0.074**	**0.071**	**0.079**	**0.081**	**0.09**	**0.088**	**0.089**
0.4	0.875	0.866	0.853	0.655	**0.066**	**0.069**	**0.071**	**0.082**	**0.085**	**0.087**	**0.087**
0.5	0.876	0.87	0.864	0.839	0.581	**0.067**	**0.067**	**0.073**	**0.079**	**0.086**	**0.085**
0.6	0.877	0.875	0.868	0.859	0.816	0.516	**0.065**	**0.067**	**0.071**	**0.08**	**0.083**
0.7	0.878	0.875	0.873	0.868	0.86	0.793	0.431	**0.065**	**0.066**	**0.075**	**0.08**
0.8	0.879	0.878	0.876	0.871	0.864	0.853	0.78	0.356	**0.065**	**0.071**	**0.076**
0.9	0.879	0.879	0.876	0.874	0.868	0.865	0.845	0.755	0.305	**0.066**	**0.073**
1	0.88	0.88	0.877	0.877	0.874	0.869	0.86	0.834	0.735	0.257	**0.064**

Table 3: **Simulated powers of the $\mathrm{GPD}_{\alpha,\lambda}$ Wald-type tests with pure data under the Poisson model (sample size 20) for various values of λ and α; the nominal level is 5%.**

$\lambda \downarrow \alpha \rightarrow$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
−0.9	0.996	0.993	0.991	0.981	0.981	0.979	0.978	0.976	0.971	0.965	0.961
−0.8	0.995	0.992	0.991	0.986	0.984	0.98	0.98	0.976	0.971	0.964	0.962
−0.7	0.998	0.994	0.992	0.989	0.987	0.984	0.976	0.973	0.97	0.969	0.964
−0.6	0.998	0.998	0.994	0.992	0.99	0.986	0.98	0.975	0.97	0.967	0.965
−0.5	0.998	0.998	0.995	0.993	0.99	0.989	0.982	0.977	0.973	0.969	0.962
−0.4	0.998	0.998	0.998	0.993	0.992	0.989	0.98	0.976	0.975	0.969	0.961
−0.3	0.998	0.998	0.998	0.994	0.992	0.99	0.983	0.978	0.971	0.969	0.961
−0.2	0.998	0.998	0.998	0.995	0.992	0.99	0.983	0.979	0.972	0.967	0.962
−0.1	0.998	0.998	0.998	0.996	0.994	0.989	0.984	0.979	0.974	0.967	0.963
0	0.998	0.998	0.998	0.996	0.993	0.99	0.984	0.982	0.976	0.969	0.962
0.1	0.997	0.998	0.998	0.995	0.995	0.99	0.986	0.982	0.978	0.97	0.961
0.2	0.995	0.998	0.998	0.995	0.995	0.99	0.988	0.984	0.98	0.97	0.965
0.3	0.996	0.997	0.997	0.995	0.994	0.992	0.99	0.986	0.981	0.97	0.966
0.4	0.994	0.994	0.996	0.995	0.994	0.992	0.992	0.987	0.982	0.973	0.968
0.5	0.992	0.993	0.996	0.994	0.994	0.992	0.992	0.99	0.984	0.977	0.968
0.6	0.991	0.991	0.994	0.993	0.993	0.993	0.992	0.992	0.984	0.978	0.971
0.7	0.989	0.992	0.992	0.992	0.992	0.993	0.992	0.99	0.987	0.981	0.971
0.8	0.988	0.991	0.99	0.99	0.992	0.993	0.992	0.99	0.988	0.982	0.976
0.9	0.986	0.989	0.99	0.991	0.99	0.992	0.992	0.99	0.988	0.985	0.98
1	0.985	0.986	0.988	0.988	0.988	0.99	0.991	0.989	0.988	0.987	0.981

Table 4: Simulated powers of the $GPD_{\alpha,\lambda}$ Wald-type tests with contaminated data under the Poisson model (sample size 20) for various values of λ and α; the nominal level is 5%.

$\lambda \downarrow \alpha \rightarrow$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
−0.9	0.995	0.994	0.991	0.985	0.979	0.978	0.976	0.974	0.965	0.956	0.946
−0.8	0.996	0.995	0.992	0.989	0.983	0.979	0.977	0.972	0.965	0.957	0.945
−0.7	0.997	0.996	0.993	0.991	0.986	0.982	0.977	0.971	0.964	0.959	0.948
−0.6	0.997	0.998	0.994	0.992	0.989	0.981	0.978	0.971	0.962	0.957	0.949
−0.5	0.993	0.999	0.996	0.992	0.991	0.985	0.979	0.971	0.963	0.957	0.946
−0.4	0.988	0.999	0.996	0.993	0.991	0.987	0.978	0.971	0.961	0.956	0.941
−0.3	0.983	0.997	0.996	0.994	0.991	0.986	0.98	0.971	0.962	0.948	0.941
−0.2	0.948	0.994	0.997	0.994	0.991	0.986	0.98	0.973	0.961	0.947	0.938
−0.1	0.827	0.99	0.997	0.995	0.991	0.986	0.98	0.976	0.959	0.947	0.936
0	0.54	0.976	0.993	0.995	0.991	0.988	0.982	0.974	0.958	0.945	0.934
0.1	0.481	0.838	0.987	0.995	0.992	0.988	0.982	0.974	0.962	0.946	0.936
0.2	0.567	0.512	0.951	0.991	0.992	0.988	0.983	0.976	0.961	0.947	0.935
0.3	0.609	0.529	0.582	0.972	0.988	0.987	0.982	0.977	0.962	0.95	0.936
0.4	0.649	0.592	0.499	0.671	0.977	0.986	0.983	0.975	0.965	0.95	0.939
0.5	0.688	0.63	0.566	0.496	0.757	0.976	0.98	0.974	0.966	0.952	0.941
0.6	0.72	0.67	0.607	0.54	0.512	0.808	0.971	0.972	0.965	0.951	0.94
0.7	0.743	0.69	0.648	0.597	0.517	0.536	0.855	0.966	0.967	0.956	0.941
0.8	0.757	0.729	0.683	0.625	0.572	0.496	0.556	0.877	0.959	0.958	0.944
0.9	0.774	0.748	0.699	0.664	0.609	0.558	0.495	0.574	0.891	0.952	0.946
1	0.786	0.76	0.732	0.686	0.643	0.597	0.541	0.501	0.586	0.901	0.949

Table 5: Simulated levels of the $GPD_{\alpha,\lambda}$ Wald-type tests with pure data under the normal model (sample size 20) for various values of λ and α; the nominal level is 5%.

$\lambda \downarrow \alpha \rightarrow$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
−0.9	0.048	0.046	0.04	0.037	0.036	0.035	0.033	0.032	0.031	0.029	0.026
−0.8	0.043	0.042	0.037	0.038	0.037	0.037	0.034	0.033	0.031	0.028	0.026
−0.7	0.04	0.041	0.037	0.039	0.038	0.037	0.035	0.034	0.034	0.029	0.028
−0.6	0.04	0.042	0.038	0.039	0.039	0.037	0.035	0.034	0.034	0.031	0.028
−0.5	0.041	0.043	0.038	0.039	0.038	0.037	0.036	0.035	0.034	0.032	0.029
−0.4	0.041	0.044	0.039	0.042	0.038	0.037	0.037	0.035	0.033	0.031	0.031
−0.3	0.041	0.044	0.041	0.043	0.038	0.037	0.037	0.035	0.033	0.031	0.032
−0.2	0.039	0.043	0.041	0.043	0.039	0.037	0.037	0.035	0.033	0.031	0.032
−0.1	0.039	0.042	0.041	0.045	0.039	0.037	0.038	0.035	0.033	0.031	0.031
0	0.039	0.041	0.041	0.045	0.039	0.037	0.038	0.035	0.033	0.031	0.031
0.1	0.039	0.039	0.04	0.045	0.04	0.037	0.037	0.035	0.034	0.031	0.031
0.2	0.039	0.039	0.039	0.045	0.039	0.037	0.037	0.035	0.034	0.031	0.031
0.3	0.039	0.039	0.039	0.042	0.04	0.037	0.037	0.034	0.034	0.032	0.03
0.4	0.04	0.039	0.038	0.041	0.039	0.037	0.037	0.034	0.034	0.032	0.03
0.5	0.04	0.039	0.039	0.039	0.039	0.037	0.037	0.034	0.034	0.032	0.03
0.6	0.042	0.039	0.039	0.038	0.038	0.037	0.038	0.034	0.034	0.031	0.03
0.7	0.044	0.039	0.039	0.037	0.035	0.037	0.038	0.034	0.034	0.031	0.029
0.8	0.047	0.042	0.04	0.036	0.034	0.036	0.038	0.034	0.034	0.031	0.029
0.9	0.05	0.044	0.04	0.037	0.033	0.035	0.037	0.035	0.034	0.031	0.029
1	0.054	0.046	0.04	0.039	0.032	0.035	0.035	0.035	0.033	0.031	0.029

Table 6: Simulated powers of the $\text{GPD}_{\alpha,\lambda}$ Wald-type tests with pure data under the normal model (sample size 20) for various values of λ and α; the nominal level is 5%.

$\lambda \downarrow \alpha \rightarrow$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
−0.9	0.998	0.998	0.998	0.995	0.991	0.985	0.985	0.983	0.98	0.976	0.972
−0.8	0.998	0.998	0.998	0.994	0.989	0.985	0.985	0.983	0.98	0.975	0.97
−0.7	0.998	0.998	0.997	0.993	0.989	0.985	0.985	0.983	0.98	0.976	0.97
−0.6	0.997	0.997	0.996	0.992	0.989	0.985	0.985	0.983	0.98	0.976	0.971
−0.5	0.997	0.997	0.995	0.992	0.99	0.985	0.985	0.983	0.982	0.976	0.971
−0.4	0.997	0.997	0.995	0.992	0.99	0.986	0.985	0.984	0.983	0.975	0.971
−0.3	0.997	0.995	0.995	0.992	0.99	0.987	0.985	0.984	0.982	0.975	0.971
−0.2	0.997	0.996	0.995	0.993	0.99	0.987	0.985	0.984	0.982	0.975	0.971
−0.1	0.996	0.996	0.995	0.994	0.992	0.987	0.985	0.984	0.982	0.975	0.971
0	0.996	0.996	0.995	0.994	0.992	0.987	0.986	0.984	0.982	0.975	0.971
0.1	0.996	0.996	0.995	0.995	0.992	0.988	0.987	0.983	0.982	0.975	0.972
0.2	0.996	0.996	0.995	0.995	0.992	0.988	0.987	0.984	0.982	0.975	0.972
0.3	0.995	0.996	0.995	0.995	0.993	0.988	0.988	0.984	0.982	0.975	0.972
0.4	0.994	0.996	0.995	0.995	0.993	0.989	0.987	0.985	0.982	0.976	0.971
0.5	0.994	0.996	0.995	0.995	0.993	0.99	0.987	0.985	0.982	0.979	0.971
0.6	0.994	0.995	0.995	0.995	0.995	0.99	0.987	0.985	0.982	0.979	0.971
0.7	0.994	0.994	0.995	0.995	0.995	0.991	0.987	0.984	0.982	0.979	0.972
0.8	0.994	0.994	0.995	0.995	0.995	0.991	0.988	0.985	0.983	0.979	0.972
0.9	0.994	0.994	0.993	0.995	0.995	0.991	0.988	0.985	0.983	0.978	0.973
1	0.994	0.994	0.993	0.995	0.995	0.991	0.988	0.985	0.983	0.979	0.974

Table 7: Simulated levels of the $\text{GPD}_{\alpha,\lambda}$ Wald-type tests with contaminated data under the normal model (sample size 20) for various values of λ and α; the nominal level is 5%.

$\lambda \downarrow \alpha \rightarrow$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
−0.9	0.073	0.069	0.069	0.067	0.067	0.062	0.057	0.055	0.048	0.046	0.046
−0.8	0.071	0.07	0.071	0.067	0.066	0.062	0.057	0.054	0.047	0.046	0.046
−0.7	0.072	0.071	0.07	0.066	0.066	0.062	0.057	0.054	0.048	0.047	0.047
−0.6	0.071	0.073	0.07	0.068	0.066	0.062	0.057	0.053	0.048	0.048	0.048
−0.5	0.071	0.073	0.069	0.067	0.065	0.062	0.057	0.053	0.049	0.048	0.048
−0.4	0.072	0.071	0.069	0.067	0.066	0.062	0.057	0.053	0.049	0.048	0.048
−0.3	0.073	0.073	0.069	0.067	0.064	0.061	0.057	0.055	0.049	0.048	0.047
−0.2	0.077	0.071	0.069	0.067	0.065	0.063	0.057	0.055	0.049	0.048	0.047
−0.1	0.12	0.081	0.07	0.067	0.066	0.063	0.057	0.055	0.049	0.048	0.047
0	0.176	0.095	0.071	0.067	0.066	0.063	0.057	0.055	0.049	0.048	0.047
0.1	0.888	0.108	0.073	0.067	0.067	0.063	0.057	0.055	0.049	0.048	0.047
0.2	0.888	0.885	0.07	0.065	0.065	0.063	0.056	0.055	0.049	0.048	0.047
0.3	0.888	0.887	0.873	0.065	0.066	0.061	0.056	0.055	0.049	0.047	0.047
0.4	0.889	0.887	0.886	0.855	0.064	0.06	0.055	0.054	0.049	0.047	0.047
0.5	0.89	0.888	0.886	0.886	0.816	0.059	0.055	0.053	0.05	0.047	0.046
0.6	0.89	0.888	0.886	0.886	0.885	0.765	0.055	0.053	0.049	0.047	0.046
0.7	0.89	0.89	0.886	0.886	0.886	0.882	0.689	0.053	0.049	0.047	0.046
0.8	0.891	0.89	0.886	0.886	0.886	0.886	0.881	0.607	0.048	0.046	0.046
0.9	0.891	0.89	0.887	0.886	0.886	0.886	0.886	0.88	0.508	0.045	0.044
1	0.891	0.891	0.888	0.886	0.885	0.886	0.886	0.886	0.872	0.409	0.044

Table 8: Simulated powers of the $GPD_{\alpha,\lambda}$ Wald-type tests with contaminated data under the normal model (sample size 20) for various values of λ and α; the nominal level is 5%.

$\lambda \downarrow \alpha \rightarrow$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
−0.9	0.985	0.986	0.982	0.978	0.977	0.973	0.969	0.963	0.961	0.954	0.947
−0.8	0.985	0.986	0.982	0.979	0.977	0.974	0.969	0.963	0.96	0.954	0.946
−0.7	0.986	0.987	0.982	0.979	0.977	0.974	0.969	0.962	0.96	0.954	0.946
−0.6	0.986	0.986	0.982	0.979	0.976	0.974	0.969	0.962	0.96	0.953	0.947
−0.5	0.986	0.986	0.982	0.979	0.976	0.974	0.968	0.962	0.96	0.953	0.947
−0.4	0.985	0.985	0.982	0.979	0.976	0.973	0.968	0.963	0.96	0.954	0.947
−0.3	0.984	0.985	0.982	0.98	0.976	0.972	0.969	0.963	0.959	0.954	0.947
−0.2	0.981	0.988	0.982	0.98	0.976	0.972	0.968	0.963	0.96	0.954	0.947
−0.1	0.958	0.988	0.981	0.98	0.976	0.972	0.968	0.963	0.96	0.955	0.948
0	0.56	0.755	0.979	0.973	0.969	0.968	0.96	0.004	0.958	0.951	0.946
0.1	0.523	0.695	0.977	0.967	0.963	0.965	0.959	0.956	0.956	0.946	0.945
0.2	0.562	0.521	0.918	0.964	0.96	0.957	0.962	0.961	0.957	0.95	0.948
0.3	0.626	0.486	0.599	0.969	0.967	0.958	0.962	0.959	0.956	0.952	0.941
0.4	0.651	0.484	0.491	0.707	0.967	0.969	0.964	0.959	0.955	0.95	0.947
0.5	0.673	0.528	0.449	0.534	0.791	0.969	0.959	0.962	0.955	0.95	0.948
0.6	0.692	0.569	0.439	0.437	0.602	0.841	0.965	0.959	0.958	0.95	0.947
0.7	0.716	0.618	0.463	0.416	0.464	0.67	0.876	0.956	0.955	0.948	0.944
0.8	0.737	0.646	0.5	0.409	0.407	0.502	0.697	0.885	0.957	0.948	0.946
0.9	0.749	0.663	0.546	0.423	0.388	0.423	0.558	0.742	0.902	0.947	0.947
1	0.771	0.688	0.579	0.454	0.386	0.372	0.444	0.598	0.765	0.905	0.949

Table 9: The Drosophila data

	Recessive lethal count					
Values	0	1	2	3	4	≥ 5
Observed Frequency	23	3	0	1	1	0

Table 10: Calculated p-value of the $GPD_{\alpha,\lambda}$ Wald-type tests using the drosophila data (full data)

$\lambda \downarrow \alpha \rightarrow$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
−0.9	0.39	0.382	0.521	0.681	0.829	0.958	0.932	0.84	0.761	0.696	0.641
−0.8	0.694	0.622	0.742	0.884	0.994	0.893	0.81	0.741	0.684	0.635	0.593
−0.7	0.878	0.738	0.838	0.967	0.923	0.835	0.763	0.703	0.653	0.611	0.574
−0.6	0.974	0.806	0.888	0.992	0.887	0.804	0.737	0.683	0.637	0.599	0.565
−0.5	0.82	0.855	0.913	0.97	0.868	0.787	0.723	0.671	0.627	0.591	0.558
−0.4	0.63	0.905	0.926	0.959	0.857	0.777	0.714	0.663	0.621	0.585	0.554
−0.3	0.401	0.98	0.936	0.956	0.852	0.771	0.708	0.658	0.616	0.581	0.55
−0.2	0.193	0.873	0.953	0.956	0.852	0.769	0.705	0.654	0.612	0.578	0.547
−0.1	0.076	0.608	0.999	0.955	0.855	0.77	0.704	0.652	0.61	0.575	0.545
0	0.029	0.301	0.869	0.94	0.857	0.771	0.704	0.651	0.608	0.573	0.543
0.1	0.012	0.118	0.605	0.884	0.852	0.773	0.705	0.651	0.607	0.572	0.541
0.2	0.006	0.046	0.309	0.737	0.825	0.772	0.707	0.651	0.607	0.57	0.54
0.3	0.003	0.019	0.134	0.49	0.747	0.758	0.706	0.652	0.607	0.57	0.539
0.4	0.002	0.009	0.058	0.261	0.584	0.715	0.698	0.651	0.607	0.569	0.538
0.5	0.001	0.005	0.027	0.127	0.377	0.614	0.672	0.647	0.606	0.569	0.537
0.6	0.001	0.003	0.014	0.063	0.213	0.456	0.61	0.631	0.603	0.568	0.536
0.7	0	0.002	0.008	0.033	0.115	0.294	0.499	0.593	0.593	0.566	0.536
0.8	0	0.001	0.004	0.018	0.064	0.176	0.359	0.516	0.569	0.559	0.533
0.9	0	0.001	0.003	0.011	0.037	0.104	0.237	0.406	0.518	0.543	0.529
1	0	0.001	0.002	0.007	0.022	0.063	0.15	0.29	0.435	0.51	0.519

Table 11: Calculated p-value of the $GPD_{\alpha,\lambda}$ Wald-type tests using the drosophila data (outliers deleted data)

$\lambda\downarrow \alpha\rightarrow$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
−0.9	0.382	0.477	0.576	0.674	0.764	0.843	0.911	0.969	0.984	0.947	0.918
−0.8	0.649	0.732	0.811	0.881	0.94	0.991	0.968	0.935	0.909	0.89	0.875
−0.7	0.789	0.857	0.918	0.971	0.985	0.949	0.92	0.898	0.881	0.868	0.859
−0.6	0.874	0.931	0.981	0.978	0.943	0.915	0.894	0.877	0.865	0.857	0.851
−0.5	0.932	0.98	0.978	0.944	0.916	0.895	0.878	0.865	0.857	0.849	0.846
−0.4	0.973	0.984	0.949	0.921	0.898	0.881	0.866	0.856	0.85	0.845	0.842
−0.3	0.995	0.959	0.929	0.904	0.884	0.87	0.859	0.85	0.845	0.842	0.84
−0.2	0.971	0.939	0.912	0.891	0.874	0.862	0.853	0.846	0.842	0.839	0.838
−0.1	0.951	0.922	0.899	0.88	0.866	0.855	0.847	0.842	0.839	0.837	0.836
0	0.936	0.909	0.888	0.872	0.859	0.85	0.844	0.839	0.837	0.836	0.835
0.1	0.922	0.899	0.879	0.865	0.854	0.846	0.841	0.837	0.835	0.834	0.835
0.2	0.91	0.889	0.872	0.859	0.85	0.842	0.838	0.835	0.834	0.833	0.834
0.3	0.901	0.88	0.865	0.854	0.845	0.84	0.836	0.833	0.832	0.832	0.833
0.4	0.892	0.874	0.86	0.849	0.842	0.837	0.833	0.832	0.831	0.831	0.832
0.5	0.885	0.868	0.855	0.846	0.839	0.835	0.831	0.83	0.83	0.831	0.832
0.6	0.879	0.863	0.851	0.842	0.837	0.833	0.83	0.829	0.829	0.83	0.831
0.7	0.873	0.858	0.847	0.84	0.834	0.83	0.829	0.828	0.829	0.829	0.831
0.8	0.868	0.853	0.844	0.836	0.832	0.829	0.827	0.827	0.828	0.829	0.83
0.9	0.863	0.85	0.841	0.834	0.83	0.828	0.826	0.826	0.827	0.828	0.83
1	0.859	0.846	0.837	0.832	0.829	0.826	0.825	0.825	0.826	0.827	0.83

Table 12: The Telephone Fault Data

Pair	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Difference	−988	−135	−78	3	59	83	93	110	189	197	204	229	289	310

Table 13: Calculated p-value of the GPD Wald-type tests using the telephone fault data (full data)

$\lambda\downarrow \alpha\rightarrow$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
−0.9	0	0	0.001	0.001	0.001	0.002	0.002	0.003	0.004	0.005	0.006
−0.8	0.001	0	0.001	0.001	0.001	0.002	0.002	0.003	0.004	0.005	0.006
−0.7	0.001	0	0.001	0.001	0.001	0.002	0.002	0.003	0.004	0.005	0.006
−0.6	0.001	0	0.001	0.001	0.001	0.002	0.002	0.003	0.004	0.005	0.006
−0.5	0.001	0	0.001	0.001	0.001	0.002	0.002	0.003	0.004	0.005	0.006
−0.4	0.001	0	0.001	0.001	0.001	0.002	0.002	0.003	0.004	0.005	0.006
−0.3	0.001	0	0.001	0.001	0.001	0.002	0.002	0.003	0.004	0.005	0.006
−0.2	0.002	0	0	0.001	0.001	0.002	0.002	0.003	0.004	0.005	0.006
−0.1	0	0	0	0.001	0.001	0.002	0.002	0.003	0.004	0.005	0.006
0	0	0	0	0.001	0.001	0.002	0.002	0.003	0.004	0.005	0.006
0.1	0.774	0.21	0	0.001	0.001	0.002	0.002	0.003	0.004	0.005	0.006
0.2	0.865	0.518	0.002	0	0.001	0.002	0.002	0.003	0.004	0.005	0.006
0.3	0.928	0.665	0.275	0.001	0.001	0.002	0.002	0.003	0.004	0.005	0.006
0.4	0.974	0.759	0.479	0.133	0.001	0.001	0.002	0.003	0.004	0.005	0.006
0.5	0.991	0.826	0.604	0.333	0.077	0.002	0.002	0.003	0.004	0.005	0.006
0.6	0.963	0.877	0.691	0.475	0.24	0.053	0.003	0.003	0.004	0.005	0.006
0.7	0.941	0.918	0.757	0.576	0.378	0.184	0.041	0.003	0.004	0.005	0.006
0.8	0.922	0.951	0.809	0.653	0.484	0.311	0.149	0.033	0.004	0.005	0.006
0.9	0.907	0.978	0.851	0.713	0.567	0.415	0.264	0.125	0.028	0.005	0.006
1	0.894	0.999	0.886	0.763	0.634	0.499	0.364	0.23	0.108	0.025	0.006

Table 14: Calculated p-value of the GPD Wald-type tests using the telephone fault data (outliers deleted data)

$\lambda \downarrow \alpha \rightarrow$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
−0.9	0.001	0.001	0.001	0.001	0.001	0.001	0.002	0.002	0.003	0.003	0.004
−0.8	0.001	0.001	0.001	0.001	0.001	0.001	0.002	0.002	0.003	0.003	0.004
−0.7	0.001	0.001	0.001	0.001	0.001	0.002	0.002	0.002	0.003	0.003	0.004
−0.6	0.001	0.001	0.001	0.001	0.001	0.002	0.002	0.002	0.003	0.003	0.004
−0.5	0.001	0.001	0.001	0.001	0.001	0.002	0.002	0.002	0.003	0.003	0.004
−0.4	0.001	0.001	0.001	0.001	0.001	0.002	0.002	0.002	0.003	0.003	0.004
−0.3	0.001	0.001	0.001	0.001	0.001	0.002	0.002	0.002	0.003	0.003	0.004
−0.2	0.001	0.001	0.001	0.001	0.001	0.002	0.002	0.002	0.003	0.003	0.004
−0.1	0.001	0.001	0.001	0.001	0.001	0.002	0.002	0.002	0.003	0.003	0.004
0	0.001	0.001	0.001	0.001	0.001	0.002	0.002	0.002	0.003	0.003	0.004
0.1	0.001	0.001	0.001	0.001	0.001	0.002	0.002	0.003	0.003	0.003	0.004
0.2	0.001	0.001	0.001	0.001	0.001	0.002	0.002	0.003	0.003	0.004	0.004
0.3	0.001	0.001	0.001	0.001	0.002	0.002	0.002	0.003	0.003	0.004	0.004
0.4	0.001	0.001	0.001	0.001	0.002	0.002	0.002	0.003	0.003	0.004	0.004
0.5	0.001	0.001	0.001	0.001	0.002	0.002	0.002	0.003	0.003	0.004	0.004
0.6	0.001	0.001	0.001	0.001	0.002	0.002	0.002	0.003	0.003	0.004	0.004
0.7	0.001	0.001	0.001	0.001	0.002	0.002	0.002	0.003	0.003	0.004	0.004
0.8	0.001	0.001	0.001	0.001	0.002	0.002	0.002	0.003	0.003	0.004	0.004
0.9	0.001	0.001	0.001	0.001	0.002	0.002	0.002	0.003	0.003	0.004	0.004
1	0.001	0.001	0.001	0.001	0.002	0.002	0.002	0.003	0.003	0.004	0.004

Table 15: Darwin's Plant Fertilization Data

Pair	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Difference	−67	−48	6	8	14	16	23	24	28	29	41	49	56	60	75

Table 16: Calculated p-value of the GPD Wald-type tests using Darwin's plant fertilization data (full data)

$\lambda \downarrow \alpha \rightarrow$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
−0.9	0.001	0	0	0	0	0	0	0	0	0	0
−0.8	0.002	0	0	0	0	0	0	0	0	0	0
−0.7	0.003	0	0	0	0	0	0	0	0	0	0
−0.6	0.005	0	0	0	0	0	0	0	0	0	0
−0.5	0.007	0.001	0	0	0	0	0	0	0	0	0
−0.4	0.01	0.003	0	0	0	0	0	0	0	0	0
−0.3	0.012	0.004	0	0	0	0	0	0	0	0	0
−0.2	0.015	0.007	0.001	0	0	0	0	0	0	0	0
−0.1	0.017	0.009	0.003	0	0	0	0	0	0	0	0
0	0.020	0.011	0.005	0	0	0	0	0	0	0	0
0.1	0.023	0.014	0.007	0.002	0	0	0	0	0	0	0
0.2	0.025	0.016	0.009	0.003	0	0	0	0	0	0	0
0.3	0.028	0.019	0.011	0.005	0.001	0	0	0	0	0	0
0.4	0.031	0.021	0.014	0.007	0.003	0	0	0	0	0	0
0.5	0.034	0.024	0.016	0.01	0.004	0.001	0	0	0	0	0
0.6	0.037	0.027	0.019	0.012	0.006	0.002	0.001	0	0	0	0
0.7	0.039	0.03	0.022	0.015	0.009	0.004	0.001	0	0	0	0
0.8	0.042	0.032	0.024	0.017	0.011	0.006	0.003	0.001	0	0	0
0.9	0.045	0.035	0.027	0.02	0.014	0.008	0.004	0.002	0.001	0	0
1	0.048	0.038	0.03	0.023	0.017	0.011	0.006	0.003	0.001	0.001	0

Table 17: Calculated p-value of the GPD Wald-type tests using Darwin's plant fertilization data (outliers deleted data)

$\lambda \downarrow \alpha \rightarrow$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
−0.9	0	0	0	0	0	0	0	0	0	0	0
−0.8	0	0	0	0	0	0	0	0	0	0	0
−0.7	0	0	0	0	0	0	0	0	0	0	0
−0.6	0	0	0	0	0	0	0	0	0	0	0
−0.5	0	0	0	0	0	0	0	0	0	0	0
−0.4	0	0	0	0	0	0	0	0	0	0	0
−0.3	0	0	0	0	0	0	0	0	0	0	0
−0.2	0	0	0	0	0	0	0	0	0	0	0
−0.1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0.1	0	0	0	0	0	0	0	0	0	0	0
0.2	0	0	0	0	0	0	0	0	0	0	0
0.3	0	0	0	0	0	0	0	0	0	0	0
0.4	0	0	0	0	0	0	0	0	0	0	0
0.5	0	0	0	0	0	0	0	0	0	0	0
0.6	0	0	0	0	0	0	0	0	0	0	0
0.7	0	0	0	0	0	0	0	0	0	0	0
0.8	0	0	0	0	0	0	0	0	0	0	0
0.9	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0