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Construction of Bivariate Fan Chart from Joint Distribution

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Abstract

A fan chart, which depicts uncertainties in projections of macroeconomic variables, is presently limited to univariate framework. However, to illustrate the uncertainties in forecasts of two mutually dependent variables, a bivariate framework is required. Using a joint distribution of two related variables, this paper proposes a theory for constructing bivariate fan chart and conditional fan chart for one variable given known information on the other. Bivariate fan charts for inflation and growth projections presented in the numerical section, show the uncertainties of both the variables in one framework. As compared to the univariate fan chart, the conditional fan chart for growth given prior information on inflation, displays noticeably narrowed confidence bands and reduced error for the revised growth projections.

Construction of Bivariate Fan Chart from Joint Distribution

Introduction

Essential inputs to a forward-looking monetary policy formulation include the plausible future trajectory or the short and long-run forecasts of various macroeconomic variables (Smaghi, 2006; McDermot, 2013). As forecasts are inevitably associated with risks[2] due to interaction of the target variable with other economic variables, the plausible risks to forecasts are required to be minimized to the extent possible. Confidence interval around a forecast value quantifies the plausible risk to the forecast. The Bank of England was the first central bank to publish uncertainty around the inflation forecasts in the form of fan chart in February 1996. Thereafter, fan chart is being used globally by most of the central banks. The Reserve Bank of India (RBI) adopted the fan chart in its Monetary Policy Statements in 2009. Fan chart is a distribution of the upside and downside risks surrounding the forecast at each horizon, constructed using a baseline forecast, the balance of risk to the forecast and the uncertainty value. The upside and downside risks to forecast may not be symmetric. In other words, the target variable realising a value higher than its forecast may have a larger probability than realising a value lower than the forecast or *vice versa*.

The theory on incorporating the asymmetric risks to forecasts were contributed by Blix and Sellin (1998) using a two-piece normal distribution. Its application in the Indian context was discussed explicitly by Banerjee and Das (2011). They discussed the use of fan chart in other countries not only in the context of forecasting inflation and growth but also in the study of survival rates. A methodology for constructing fan chart for government deficit and debt ratios over medium-term was discussed by Cronin and Dowd (2011). On the other hand, Cogley *et al*. (2005), Österholm (2008) and Franta *et al*. (2011) researched on Bayesian fan charts. Other contributions in the area of fan charts include those of Österholm (2006), Kannan and Elekdag (2009), Pońsko and Rybaczyk (2016), Razi and Loke (2017) and Turner (2017) among others. Recently, Turner *et al*. (2018) proved that quantification of skewness in fan charts towards downside or upside risks to forecasts by assessing the probability of a future dip using a probit model is useful.

Although, among the macroeconomic variables of interest in the policy formulation, there are variables which are inter-dependent, the literature contains

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[2] The term 'risk' in this paper is not related to the variable; rather, it is related to the forecasts of the variable. Risk to forecast signifies the uncertainty associated with the forecast that the target variable will realise the forecast value in the time point for which the forecast was made. Both upside and downside deviation of the realised value from its forecast are considered as 'risk to forecast'.

limited work on dimensional extension of fan charts. A derivation of the bivariate fan chart of inflation and output forecasts was attempted by Blix and Sellin (2000). The paper opines that a bivariate framework allows for forecast of one variable conditional on information on the other. The paper clarifies the reasons behind not using an econometric model for deriving confidence bands. Firstly, multiple models are used for deriving inflation and growth forecasts. Secondly, specific information concerning a particular forecast period cannot be incorporated into the model. Lastly, important subjective judgments need to be taken into account as rigorously as possible for obtaining good short-term forecasts. The paper assumes that error distribution of inflation and output forecasts are separately two-piece normal and links the standardised form of these variables into a standard Bivariate normal (BN) distribution through an estimated correlation coefficient. The paper derives conditional distribution of one variable given the other. However, as the two variables of interest are mutually related, an initial assumption of joint distribution of the two variables would give more insights on their trajectory. The author assumes a bivariate relationship at the first stage using a joint distribution and then deriving the conditional fan chart.

The dimensional extension of fan charts is important because there exists mutually related variables in the economy, the forecasts of which form important inputs to the monetary policy formation. The relationship between two variables can be incorporated in the forecasting model itself. However, the relationship should also be visible in the error distribution surrounding their forecasts. The information on risk to forecast of one variable affecting the risk to forecast of the other is important from the policy perspective. As fan charts are error distribution around the forecast, a bivariate fan chart would basically mean the joint error distribution around the two-dimensional forecast coordinates. Thus, the bivariate fan chart is not a model; rather an error visualisation around the forecast coordinates which helps in constructing a conditional fan chart of one variable given information on error committed in forecasting one variable and then reducing the error which will be committed in forecasting the other. This information will particularly be helpful when the information on the variables is available with different lags.

Section II discusses about the theoretical background of the existing univariate fan chart. A theory of bivariate fan chart construction involving two related variables and a conditional fan chart for one variable given known information on the other is discussed in the Section III. Numerical demonstration of the performance of bivariate and conditional fan charts, based on two economic variables of interest, is presented in Section IV followed by the concluding remarks.

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II. Univariate Fan Chart

The univariate fan chart methodology discussed in the Indian context in this section largely draws upon Banerjee and Das (2011). The basic idea is to incorporate asymmetry in the upside and downside risks to the forecast of a variable, say X. To accommodate this, a two-piece normal distribution is used as given in Johnson *et al.* (1994), in which half pieces of two different normal distributions with the same mode μ but with different standard deviations σ_1 and σ_2 are joined together. The mode μ is the forecast of X and an output of some forecasting model(s) using various macroeconomic indicators.

$$f(x) = \begin{cases} C \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma_1}\right)^2\right\}, if -\infty < x \leq \mu, \\ C \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma_2}\right)^2\right\}, if\ x > \mu \end{cases} , C = \frac{\sqrt{2}}{\sqrt{\pi}(\sigma_1 + \sigma_2)}, -\infty < \mu < \infty,\ \sigma_1, \sigma_2 > 0$$

(1)

The balance of risk, which indicates the risk on the upward direction relative to the total risks to the forecast is:

$$p = P(X \leq \mu) = \frac{\sigma_1}{\sigma_1 + \sigma_2}$$

(2)

The αth quantile of this distribution is:

$$k = \begin{cases} \mu + \sigma_1 \Phi^{-1}\left(\frac{\alpha}{C\sqrt{2\pi}\sigma_1}\right), & ,if\ \alpha \leq p \\ \mu + \sigma_2 \Phi^{-1}\left(\frac{\alpha + C\sqrt{2\pi}\sigma_2 - 1}{C\sqrt{2\pi}\sigma_2}\right), if\ \alpha > p \end{cases}, \quad 0 \leq \alpha \leq 1$$

(3)

Quantiles for different values of α form the confidence intervals around the forecast μ. To find the values of σ_1 and σ_2, an alternate parameterisation of the distribution is considered as below:

$$f(x) = \begin{cases} \frac{A}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1-\gamma}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}, if -\infty < x \leq \mu \\ \frac{A}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1+\gamma}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}, & if\ x > \mu \end{cases}, \quad A = \frac{2}{\frac{1}{\sqrt{1-\gamma}} + \frac{1}{\sqrt{1+\gamma}}}$$

(4)

$-1 < \gamma < 1$, is the inverse skewness indicator and σ is uncertainty value. Comparing this with (1),

$$\sigma_1 = \frac{\sigma}{\sqrt{1-\gamma}}, \quad \sigma_2 = \frac{\sigma}{\sqrt{1+\gamma}}$$

(5)

If $\gamma > 0$, $\sigma_1 > \sigma_2$, the distribution is biased to the left, if $\gamma < 0$, $\sigma_1 < \sigma_2$, the distribution is biased to the right and if $\gamma = 0$, $\sigma_1 = \sigma_2$, the distribution is symmetric. Using (2) and (5), we get the following expression for γ,

$$\gamma = \frac{2p - 1}{1 - 2p + 2p^2} \tag{6}$$

Value of γ can be obtained after determining the value of p as per judgement about the perception of future scenario of the target variable. Putting this value γ in (5) and taking σ as the root mean square of historical deviations of the forecasts from their realised values, σ_1 and σ_2 can be easily calculated. Next, using (3), quantiles are obtained which form the confidence bands in the fan chart. Thus, the values of three parameters *viz.*, baseline forecast μ, uncertainty σ and balance of risk p are required to construct a univarite fan chart. It may be noted that better the performances of historical forecasts, smaller the values of σ_1 and σ_2 and hence, narrower the confidence intervals.

III. Bivariate and Conditional Fan Charts

III.a. Bivariate Fan Chart from Joint Distribution

Consider two variables X and Y and their forecast values for time point t as, μ_x and μ_y, respectively. With the arguments similar to those for univariate fan chart in Section II, it may be presumed that possible outcomes for X and Y will symmetrically disperse around a central coordinate or the most probable coordinate and they will more likely be closer to the central coordinates than those further away. Hence, a three-dimensional bell-shaped distribution *i.e.*, a bivariate normal distribution can be assumed as the joint error distribution of the forecasts. Similar to the univariate case, (μ_x, μ_y) is considered as the modal coordinate of the bivariate normal distribution. The modal coordinate uses less information about the distribution and it is less affected by outliers and other modes in the distribution, if multimodal. However, the bivariate normal is neither multimodal nor too flat, hence the above issues are not of concern. Also, asymptotic properties are required in case of inference problems where the finite-sample distribution is unknown; however, in the present scenario, the distribution is known.

Assuming the origin as (μ_x, μ_y), the risk associated to forecast coordinates (μ_x, μ_y) may be not be equal in all the four quadrants of the two-dimensional plane.

Bias in the first quadrant implies that the realised values for both X and Y may be higher than their forecasts. Bias in the 2nd quadrant indicates that X may realise a value less than its forecast but risk for Y is towards values higher than its forecast. Similarly, 3rd quadrant bias signifies that there is a risk for both X and Y to realise values lower than their forecasts and bias in the 4th quadrant hints that X may realise value higher than its forecast and Y, a value lower than its forecast. The unbalanced risk associated with the forecast coordinate or the asymmetry in the plausible deviations need to be incorporated in the distribution itself. An approach similar to the univariate framework is required in the bivariate case *i.e.*, joining half pieces of two bivariate normal distributions with same modal coordinate (μ_x, μ_y), where $-\infty < \mu_x, \mu_y < \infty$ and with the same correlation coefficient ρ, $-1 \leq \rho \leq 1$ but with different standard deviations that is $\sigma_{1x}(>0)$ and $\sigma_{1y}(>0)$ for (X,Y) in the first distribution and $\sigma_{2x}(>0)$ and $\sigma_{2y}(>0)$ for (X,Y) in the second distribution to incorporate the unbalanced risks in the four quadrants. It is observed that the literature including Kotz *et al.* (2000), Balakrishnan and Lai (2009) contain theory on truncated bivariate normal distributions, mixtures of bivariate normal distributions, bivariate half-normal distributions (some or all of the variates in a bivariate normal are replaced by their absolute values) and bivariate skew-normal distribution (to control for skewness in bivariate normal distributions). However, as theory on two-piece bivariate normal distribution does not exist, an attempt has been made to construct the same.

The bivariate distribution of (X,Y) can be visualised in a three-dimensional set-up, *viz.* $x \in \mathbb{R}$, $y \in \mathbb{R}$ and z for probability, $0 \leq z \leq 1$. The next step is to divide the distribution into two pieces using a straight line (or the cut line). To generalise the straight line, an equation $(y - \mu_y) = m(x - \mu_x)$, $x \in \mathbb{R}, y \in \mathbb{R}$ may be considered, with $m\,(\in \mathbb{R})$ as the slope of the line. Hence, using the straight line $(y - \mu_y) \leq m(x - \mu_x)$, one piece of the bivariate normal $BN(\mu_x, \mu_y, \sigma_{1x}, \sigma_{1y}, \rho)$ and another piece of the bivariate normal $BN(\mu_x, \mu_y, \sigma_{2x}, \sigma_{2y}, \rho)$ can be joined as below:

$$
f(x,y) = \begin{cases} C \exp\left[-\dfrac{1}{2(1-\rho^2)}\left\{\left(\dfrac{x-\mu_x}{\sigma_{1x}}\right)^2 - 2\rho\left(\dfrac{x-\mu_x}{\sigma_{1x}}\right)\left(\dfrac{y-\mu_y}{\sigma_{1y}}\right) + \left(\dfrac{y-\mu_y}{\sigma_{1y}}\right)^2\right\}\right], \\ \qquad\qquad if\ (y-\mu_y) \leq m(x-\mu_x), \quad x \in \mathbb{R}, y \in \mathbb{R} \\[4pt] C \exp\left[-\dfrac{1}{2(1-\rho^2)}\left\{\left(\dfrac{x-\mu_x}{\sigma_{2x}}\right)^2 - 2\rho\left(\dfrac{x-\mu_x}{\sigma_{2x}}\right)\left(\dfrac{y-\mu_y}{\sigma_{2y}}\right) + \left(\dfrac{y-\mu_y}{\sigma_{2y}}\right)^2\right\}\right], \\ \qquad\qquad if\ (y-\mu_y) > m(x-\mu_x), \quad x \in \mathbb{R}, y \in \mathbb{R} \end{cases}
$$

(7)

Here, C is a constant such that,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1$$

which gives,

$C =$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}\left[\sigma_{1x}\sigma_{1y}\left\{\frac{1}{2}-\frac{1}{2\pi}tan^{-1}\left(\frac{m\frac{\sigma_{1x}}{\sigma_{1y}}-\rho}{\sqrt{1-\rho^2}}\right)-\frac{1}{2\pi}tan^{-1}\left(\frac{m\frac{\sigma_{1x}}{\sigma_{1y}}+\rho}{\sqrt{1-\rho^2}}\right)\right\}+\sigma_{2x}\sigma_{2y}\left\{\frac{1}{2}-\frac{1}{2\pi}tan^{-1}\left(\frac{m\frac{\sigma_{2x}}{\sigma_{2y}}-\rho}{\sqrt{1-\rho^2}}\right)-\frac{1}{2\pi}tan^{-1}\left(\frac{m\frac{\sigma_{2x}}{\sigma_{2y}}+\rho}{\sqrt{1-\rho^2}}\right)\right\}\right]}$$

As μ_x and μ_y are forecasts of X and Y respectively, hence, the following obvious assumptions follow:

$$E(X) = \mu_x \quad and \quad E(Y) = \mu_y \tag{8}$$

These assumptions can be successfully applied for time period t, if both the error series *viz.*, $e_{xi} = (x_i - \mu_{xi})$ for X and $e_{yi} = (y_i - \mu_{yi})$ for Y, with $i = 1,2,...,(t-1)$, are standard normal, where,

μ_{xi} and μ_{yi} = forecasts of X and Y, respectively, for period i;

x_i and y_i = realised values of X and Y, respectively, at period i.

Putting $m = 0$, $\sigma_{1x} = \sigma_{2x}$ and $\sigma_{1y} = \sigma_{2y}$ in (7), $f(x,y)$ reduces to a bivariate normal distribution. The joint balance of risk (BOR) indicating the relative risks on various directions of the forecast coordinate (μ_x,μ_y), relative to the total risk, can be defined as given below:

$$p(1) = P\{X > \mu_x, Y > \mu_y + m(x-\mu_x)\} = C\sigma_{2x}\sigma_{2y}2\pi\sqrt{1-\rho^2}\left\{\frac{1}{4}-\frac{1}{2\pi}tan^{-1}\left(\frac{m\frac{\sigma_{2x}}{\sigma_{2y}}-\rho}{\sqrt{1-\rho^2}}\right)\right\}$$

$$p(2) = P\{X \le \mu_x, Y > \mu_y + m(x-\mu_x)\} = C\sigma_{2x}\sigma_{2y}2\pi\sqrt{1-\rho^2}\left\{\frac{1}{4}-\frac{1}{2\pi}tan^{-1}\left(\frac{m\frac{\sigma_{2x}}{\sigma_{2y}}+\rho}{\sqrt{1-\rho^2}}\right)\right\}$$

$$p(3) = P\{X \le \mu_x, Y \le \mu_y + m(x-\mu_x)\} = C\sigma_{1x}\sigma_{1y}2\pi\sqrt{1-\rho^2}\left\{\frac{1}{4}-\frac{1}{2\pi}tan^{-1}\left(\frac{m\frac{\sigma_{1x}}{\sigma_{1y}}-\rho}{\sqrt{1-\rho^2}}\right)\right\}$$

$$p(4) = P\{X > \mu_x, Y \le \mu_y + m(x-\mu_x)\} = C\sigma_{1x}\sigma_{1y}2\pi\sqrt{1-\rho^2}\left\{\frac{1}{4}-\frac{1}{2\pi}tan^{-1}\left(\frac{m\frac{\sigma_{1x}}{\sigma_{1y}}+\rho}{\sqrt{1-\rho^2}}\right)\right\}$$

$$\tag{9}$$

Risk to forecast coordinates will be the highest in the direction with the highest BOR. An alternate parametrisation for $f(x,y)$ in (7) can be written as:

$$f(x,y) =$$

$$\begin{cases} \frac{D}{2\pi\sqrt{1-\rho^2}\sigma_x\sigma_y} exp\left[-\frac{1}{2(1-\rho^2)}\left\{(1-\gamma_x)\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\sqrt{(1-\gamma_x)(1-\gamma_y)}\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + (1-\gamma_y)\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right\}\right] \\ \qquad if \ (y-\mu_y) \le m(x-\mu_x), \ y \in \mathbb{R}, x \in \mathbb{R} \\ \\ \frac{D}{2\pi\sqrt{1-\rho^2}\sigma_x\sigma_y} exp\left[-\frac{1}{2(1-\rho^2)}\left\{(1+\gamma_x)\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\sqrt{(1+\gamma_x)(1+\gamma_y)}\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + (1+\gamma_y)\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right\}\right] \\ \qquad if \ (y-\mu_y) > m(x-\mu_x), \ y \in \mathbb{R}, x \in \mathbb{R} \end{cases}$$

$$(10)$$

with γ_x and γ_y as the inverse skewness indicators of X and Y, respectively, σ_x and σ_y as the uncertainty values of X and Y, respectively, and D as a constant such that,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1 \qquad \text{providing,}$$

$$D = \frac{1}{\frac{1}{\sqrt{(1-\gamma_x)(1-\gamma_y)}}\left\{\left[\frac{1}{2}-\frac{1}{2\pi}tan^{-1}\left(\frac{m\frac{\sigma_x\sqrt{1-\gamma_y}}{\sigma_y\sqrt{1-\gamma_x}}-\rho}{\sqrt{1-\rho^2}}\right)\right]-\frac{1}{2\pi}tan^{-1}\left(\frac{m\frac{\sigma_x\sqrt{1-\gamma_y}}{\sigma_y\sqrt{1-\gamma_x}}+\rho}{\sqrt{1-\rho^2}}\right)\right\}+\frac{1}{\sqrt{(1+\gamma_x)(1+\gamma_y)}}\left\{\left[\frac{1}{2}-\frac{1}{2\pi}tan^{-1}\left(\frac{m\frac{\sigma_x\sqrt{1+\gamma_y}}{\sigma_y\sqrt{1+\gamma_x}}-\rho}{\sqrt{1-\rho^2}}\right)\right]-\frac{1}{2\pi}tan^{-1}\left(\frac{m\frac{\sigma_x\sqrt{1+\gamma_y}}{\sigma_y\sqrt{1+\gamma_x}}+\rho}{\sqrt{1-\rho^2}}\right)\right\}}$$

If $m = 0, \gamma_x = \gamma_y = 0$, then the distribution in (10) reduces to a bivariate normal distribution. Establishing a parametric relationship between (7) and (10),

$$\sigma_{1x} = \frac{\sigma_x}{\sqrt{1-\gamma_x}}, \qquad \sigma_{1y} = \frac{\sigma_y}{\sqrt{1-\gamma_y}}, \qquad \sigma_{2x} = \frac{\sigma_x}{\sqrt{1+\gamma_x}}, \qquad \sigma_{2y} = \frac{\sigma_y}{\sqrt{1+\gamma_y}} \qquad (11)$$

The values of balance of risks *i.e.*, $p_x = \frac{\sigma_{1x}}{\sigma_{1x}+\sigma_{2x}}$ and $p_y = \frac{\sigma_{1y}}{\sigma_{1y}+\sigma_{2y}}$ are either separately derived as per judgements about the perception of the future scenario of X and Y, respectively, or derived on the basis of data[3]. Using the assumptions in (8),

$$\sigma_x = \sqrt{E(X-\mu_x)^2} \quad and \quad \sigma_y = \sqrt{E(Y-\mu_y)^2} \qquad (12)$$

The proxy for σ_x and σ_y are taken as σ_x^* and σ_y^*, the root mean square of historical deviations of the forecasts from their respective realised values, as given below:

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[3] Methodologies for modelling the skewness of fan chart distribution are given by Österholm (2006) and Turner *et al.* (2018).

$$\sigma_x^* = \sqrt{\frac{1}{t-1}\sum_{i=1}^{t-1}(x_i - \mu_{xi})^2} \ \ for \ \sigma_x \ \ and \ \ \sigma_y^* = \sqrt{\frac{1}{t-1}\sum_{i=1}^{t-1}(y_i - \mu_{yi})^2} \ \ for \ \sigma_y$$

(13)

Better the past forecasting performances of μ_{xi}'s and μ_{yi}'s, smaller will be the values of σ_x^* and σ_y^*. Next, putting expressions for σ_{1x} and σ_{1y} in p_x,

$$p_x = \frac{1}{1+\sqrt{\frac{1-\gamma_x}{1+\gamma_x}}}, \ \ solving \ to \ \ \gamma_x = \frac{2p_x - 1}{1 - 2p_x + 2p_x^2},$$

(14)

Similarly,

$$\gamma_y = \frac{2p_y - 1}{1 - 2p_y + 2p_y^2}$$

(15)

Next, using equation 11, the values of σ_{1x}, σ_{2x}, σ_{1y} and σ_{2y} can be computed. Also, a proxy for ρ is,

$$\rho^* = \frac{\sum_{i=1}^{t-1}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{t-1}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{t-1}(y_i - \bar{y})^2}}$$

(16)

i.e., the correlation between X and Y at time period $(t-1)$.

The confidence bands for the bivariate distribution discussed above is a set of equi-probability contours[4]. Consider all coordinates (x, y) for which the density is same, say δ. Then,

$$C \ exp\left[-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_x}{\sigma_{1x}}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_{1x}}\right)\left(\frac{y-\mu_y}{\sigma_{1y}}\right) + \left(\frac{y-\mu_y}{\sigma_{1y}}\right)^2\right\}\right] = \delta,$$

$$if \ (y - \mu_y) \le m(x - \mu_x), \quad y \in \mathbb{R}, x \in \mathbb{R}$$

$$C \ exp\left[-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_x}{\sigma_{2x}}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_{2x}}\right)\left(\frac{y-\mu_y}{\sigma_{2y}}\right) + \left(\frac{y-\mu_y}{\sigma_{2y}}\right)^2\right\}\right] = \delta,$$

$$if \ (y - \mu_y) > m(x - \mu_x), \quad y \in \mathbb{R}, x \in \mathbb{R}$$

$$or, \frac{(x-\mu_x)^2}{\delta^* \sigma_{1x}^2} - 2\rho\frac{(x-\mu_x)}{\delta^* \sigma_{1x}}\frac{(y-\mu_y)}{\delta^* \sigma_{1y}} + \frac{(y-\mu_y)^2}{\delta^* \sigma_{1y}^2} = 1, \quad if \ (y - \mu_y) \le m(x - \mu_x), y \in \mathbb{R}, x \in \mathbb{R}$$

$$and, \frac{(x-\mu_x)^2}{\delta^* \sigma_{2x}^2} - 2\rho\frac{(x-\mu_x)}{\delta^* \sigma_{2x}}\frac{(y-\mu_y)}{\delta^* \sigma_{2y}} + \frac{(y-\mu_y)^2}{\delta^* \sigma_{2y}^2} = 1, \quad if \ (y - \mu_y) > m(x - \mu_x), y \in \mathbb{R}, x \in \mathbb{R}$$

(17)

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[4] An equi-probability contour joins all the coordinates with same probability or density.

$$where, \quad \delta^* = -2(1-\rho^2)ln\left(\frac{\delta}{C}\right)$$

Thus, for constructing a bivariate fan chart, one needs to know the values of seven parameters *viz.*, baseline forecasts or modes (μ_x, μ_y), the uncertainties σ_x and σ_y, the balance of risks p_x and p_y and the correlation coefficient ρ.

III.b. Conditional Fan Chart

The marginal distribution for X denoted by $g(x)$ is $g(x) = \int_{-\infty}^{\infty} f(x,y)dy$ (detailed derivation in Annexure 1) is,

$$g(x) = C\sigma_{1y}\sqrt{1-\rho^2}\sqrt{2\pi}\,exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_{1x}^2}\right\}\Phi\left\{\frac{\frac{m(x-\mu_x)}{\sigma_{1y}} - \frac{\rho(x-\mu_x)}{\sigma_{1x}}}{\sqrt{1-\rho^2}}\right\}$$

$$+ C\sigma_{2y}\sqrt{1-\rho^2}\sqrt{2\pi}\,exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_{2x}^2}\right\}\Phi\left\{-\frac{\frac{m(x-\mu_x)}{\sigma_{2y}} - \frac{\rho(x-\mu_x)}{\sigma_{2x}}}{\sqrt{1-\rho^2}}\right\}, \quad x \in \mathbb{R}$$

(18)

Given the information on variable X, revised forecast of Y and its confidence band are to be derived. Since the forecast of Y uses the available information, it is expected to be better than the original estimate μ_y. Distribution of $(Y|X = x)$ denoted by $f_Y(y|x)$ is

$$f_Y(y|x) = \frac{f(x,y)}{g(x)}$$

$$= \begin{cases} \dfrac{C\,exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_{1x}^2}\right\}exp\left[-\frac{1}{2(1-\rho^2)}\left\{\frac{y-\mu_y}{\sigma_{1y}} - \rho\left(\frac{x-\mu_x}{\sigma_{1x}}\right)\right\}^2\right]}{g(x)}, & if\ y \le \mu_y + m(x-\mu_x) \\[3em] \dfrac{C\,exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_{2x}^2}\right\}exp\left[-\frac{1}{2(1-\rho^2)}\left\{\frac{y-\mu_y}{\sigma_{2y}} - \rho\left(\frac{x-\mu_x}{\sigma_{2x}}\right)\right\}^2\right]}{g(x)}, & if\ y > \mu_y + m(x-\mu_x) \end{cases}$$

(19)

Now, for a given value of X, say x, mean of the conditional distribution is (detail derivation in Annexure 1),

$$E(Y|x) = \int_{-\infty}^{\infty} yf_Y(y|x)dy$$

$$= \frac{C\sqrt{1-\rho^2}}{g(x)} \sigma_{1y}{}^2 exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_{1x}{}^2}\right\}\left[\left[-\sqrt{2\pi}\sqrt{1-\rho^2}\ \phi\left\{\frac{\frac{m(x-\mu_x)}{\sigma_{1y}}-\frac{\rho(x-\mu_x)}{\sigma_{1x}}}{\sqrt{1-\rho^2}}\right\}\right.\right.$$

$$\left.\left.+\frac{\rho(x-\mu_x)}{\sigma_{1x}}\sqrt{2\pi}\Phi\left\{\frac{\frac{m(x-\mu_x)}{\sigma_{1y}}-\frac{\rho(x-\mu_x)}{\sigma_{1x}}}{\sqrt{1-\rho^2}}\right\}\right]\right]$$

$$+\frac{C\sqrt{1-\rho^2}}{g(x)}\sigma_{1y}\mu_y\sqrt{2\pi}\ exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_{1x}{}^2}\right\}\Phi\left\{\frac{\frac{m(x-\mu_x)}{\sigma_{1y}}-\frac{\rho(x-\mu_x)}{\sigma_{1x}}}{\sqrt{1-\rho^2}}\right\}$$

$$+\frac{C\sqrt{1-\rho^2}}{g(x)}\sigma_{2y}{}^2 exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_{2x}{}^2}\right\}\left[\left[\sqrt{2\pi}\sqrt{1-\rho^2}\ \phi\left\{\frac{\frac{m(x-\mu_x)}{\sigma_{2y}}-\frac{\rho(x-\mu_x)}{\sigma_{2x}}}{\sqrt{1-\rho^2}}\right\}\right.\right.$$

$$\left.\left.+\frac{\rho(x-\mu_x)}{\sigma_{2x}}\sqrt{2\pi}\Phi\left\{-\frac{\frac{m(x-\mu_x)}{\sigma_{2y}}-\frac{\rho(x-\mu_x)}{\sigma_{2x}}}{\sqrt{1-\rho^2}}\right\}\right]\right]$$

$$+\frac{C\sqrt{1-\rho^2}}{g(x)}\sigma_{2y}\mu_y\sqrt{2\pi}\ exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_{2x}{}^2}\right\}\Phi\left\{-\frac{\frac{m(x-\mu_x)}{\sigma_{2y}}-\frac{\rho(x-\mu_x)}{\sigma_{2x}}}{\sqrt{1-\rho^2}}\right\}$$

(20)

$E(Y|x)$ is proposed as the revised forecast for Y given known information on X.

The modes of the conditional distribution $f_Y(y|x)$ are:

$$(Y|x)_{Mode-1} = \mu_y + \rho\frac{\sigma_{1y}}{\sigma_{1x}}(x-\mu_x)\ if\ y \le \mu_y + m(x-\mu_x)$$

$$and\ (Y|x)_{Mode-2} = \mu_y + \rho\frac{\sigma_{2y}}{\sigma_{2x}}(x-\mu_x)\ if\ y > \mu_y + m(x-\mu_x)$$

Among $(Y|x)_{Mode-1}$ and $(Y|x)_{Mode-2}$, the mode of $f_Y(y|x)$ is

$$(Y|x)_{Mode}$$
$$=\begin{cases}(Y|x)_{Mode-1}, if\ max[f_y\{(Y|x)_{Mode-1}|x\}, f_y\{(Y|x)_{Mode-2}|x\}] = f_y\{(Y|x)_{Mode-1}|x\}\\(Y|x)_{Mode-2}, if\ max[f_y\{(Y|x)_{Mode-1}|x\}, f_y\{(Y|x)_{Mode-2}|x\}] = f_y\{(Y|x)_{Mode-2}|x\}\end{cases}$$
(21)

As an alternative to $E(Y|x)$, the revised forecast for Y, can be taken as $(Y|x)_{Mode}$.

The next step is to derive the revised confidence bands for Y. Now,

$$P\{Y \leq \mu_y + m(x - \mu_x)|x\}$$

$$= \frac{C}{g(x)} \int\limits_{-\infty}^{\mu_y + m(x-\mu_x)} exp\left[-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_x}{\sigma_{1x}}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_{1x}}\right)\left(\frac{y-\mu_y}{\sigma_{1y}}\right)\right.\right.$$

$$\left.\left. + \left(\frac{y-\mu_y}{\sigma_{1y}}\right)^2\right\}\right] dy$$

$$= \frac{C\sigma_{1y}\sqrt{2\pi}\sqrt{1-\rho^2}exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_{1x}^2}\right\}}{g(x)} \cdot \Phi\left\{\frac{\frac{m(x-\mu_x)}{\sigma_{1y}} - \frac{\rho(x-\mu_x)}{\sigma_{1x}}}{\sqrt{1-\rho^2}}\right\} = A \text{ (say)}$$

$$(22)$$

Now, consider, $P(Y \leq k|x) = \alpha, \ 0 \leq \alpha \leq 1$

Case-1: If $k \leq \mu_y + m(x - \mu_x)$, then, $P(Y \leq k|x) = \alpha$ gives,

$$\frac{C}{g(x)} \int\limits_{-\infty}^{k} exp\left[-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_x}{\sigma_{1x}}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_{1x}}\right)\left(\frac{y-\mu_y}{\sigma_{1y}}\right) + \left(\frac{y-\mu_y}{\sigma_{1y}}\right)^2\right\}\right] dy = \alpha$$

or, $\quad \dfrac{C\sigma_{1y}\sqrt{2\pi}\sqrt{1-\rho^2}exp\left\{-\dfrac{(x-\mu_x)^2}{2\sigma_{1x}^2}\right\}}{g(x)} \cdot \Phi\left\{\dfrac{\dfrac{k-\mu_y}{\sigma_{1y}} - \dfrac{\rho(x-\mu_x)}{\sigma_{1x}}}{\sqrt{1-\rho^2}}\right\} = \alpha$

$$(23)$$

Case-2: If $k > \mu_y + m(x - \mu_x)$, $P[Y \leq k|x] = \alpha$, gives,

$$P(Y \leq \mu_y|x) + P(Y \leq k|x) = \alpha$$

or, $A + \dfrac{C}{g(x)} \int\limits_{\mu_y + m(x-\mu_x)}^{k} exp\left[-\dfrac{1}{2(1-\rho^2)}\left\{\left(\dfrac{x-\mu_x}{\sigma_{2x}}\right)^2 - 2\rho\left(\dfrac{x-\mu_x}{\sigma_{2x}}\right)\left(\dfrac{y-\mu_y}{\sigma_{2y}}\right)\right.\right.$

$$\left.\left. + \left(\frac{y-\mu_y}{\sigma_{2y}}\right)^2\right\}\right] dy = \alpha$$

reducing to,

$$A + \cfrac{C\sigma_{2y}\sqrt{2\pi}\sqrt{1-\rho^2}\exp\left\{-\cfrac{(x-\mu_x)^2}{2\sigma_{2x}^2}\right\}}{g(x)} \left[\Phi\left\{\cfrac{\cfrac{k-\mu_y}{\sigma_{2y}} - \cfrac{\rho(x-\mu_x)}{\sigma_{2x}}}{\sqrt{1-\rho^2}}\right\}\right.$$
$$\left.- \Phi\left\{\cfrac{\cfrac{m(x-\mu_x)}{\sigma_{2y}} - \cfrac{\rho(x-\mu_x)}{\sigma_{2x}}}{\sqrt{1-\rho^2}}\right\}\right] = \alpha$$

(24)

Hence, using (23) and (24),

$$k = \begin{cases} \sigma_{1y}\left[\sqrt{1-\rho^2}\Phi^{-1}\left[\cfrac{\alpha g(x)}{C\sigma_{1y}\sqrt{2\pi}\sqrt{1-\rho^2}\exp\left\{-\cfrac{(x-\mu_x)^2}{2\sigma_{1x}^2}\right\}}\right] + \cfrac{\rho(x-\mu_x)}{\sigma_{1x}}\right] + \mu_y \\ \qquad\qquad if, \quad \alpha \le A \\[2em] \sigma_{2y}\left[\sqrt{1-\rho^2}\Phi^{-1}\left[\cfrac{(\alpha - A)g(x)}{C\sigma_{2y}\sqrt{2\pi}\sqrt{1-\rho^2}\exp\left\{-\cfrac{(x-\mu_x)^2}{2\sigma_{2x}^2}\right\}} + \Phi\left\{\cfrac{\cfrac{m(x-\mu_x)}{\sigma_{2y}} - \cfrac{\rho(x-\mu_x)}{\sigma_{2x}}}{\sqrt{1-\rho^2}}\right\}\right] + \cfrac{\rho(x-\mu_x)}{\sigma_{2x}}\right] + \mu_y \\ \qquad\qquad if, \quad \alpha > A \end{cases}$$

(25)

The quantiles computed using (25), for different values of α form the confidence intervals around the revised forecast. Due to the use of additional information on X, the revised confidence intervals are expected to be narrower than the confidence intervals for the univariate fan chart.

The balance of risk $P\{Y \le E(Y|x)|x\}$ in this case, for the conditional mean as the forecast, can be obtained by replacing k with $E(Y|x)$ in (23) and (24):

$$BOR_{Mean} = P\{Y \le E(Y|x)|x\} =$$

$$\begin{cases} \cfrac{C\sigma_{1y}\sqrt{2\pi}\sqrt{1-\rho^2}\exp\left\{-\cfrac{(x-\mu_x)^2}{2\sigma_{1x}^2}\right\}}{g(x)}\Phi\left\{\cfrac{\cfrac{E(Y|x)-\mu_y}{\sigma_{1y}} - \cfrac{\rho(x-\mu_x)}{\sigma_{1x}}}{\sqrt{1-\rho^2}}\right\}, if\ E(Y|x) \le \mu_y + m(x-\mu_x) \\[2em] A + \cfrac{C\sigma_{2y}\sqrt{2\pi}\sqrt{1-\rho^2}\exp\left\{-\cfrac{(x-\mu_x)^2}{2\sigma_{2x}^2}\right\}}{g(x)}\left[\Phi\left\{\cfrac{\cfrac{E(Y|x)-\mu_y}{\sigma_{2y}} - \cfrac{\rho(x-\mu_x)}{\sigma_{2x}}}{\sqrt{1-\rho^2}}\right\} - \Phi\left\{\cfrac{\cfrac{m(x-\mu_x)}{\sigma_{2y}} - \cfrac{\rho(x-\mu_x)}{\sigma_{2x}}}{\sqrt{1-\rho^2}}\right\}\right], if\ E(Y|x) > \mu_y + m(x-\mu_x) \end{cases}$$

(26)

Hence, the conditional fan chart will be negatively skewed and the risk to forecast will have downside bias, if $BOR_{Mean} < 0.5$ and will be positively skewed and the risk to forecast will have upside bias if $BOR_{Mean} > 0.5$ and symmetric *i.e.,* balanced risk to forecast if $BOR_{Mean} = 0.5$.

The balance of risk $P\{Y \le (Y|x)_{Mode}|x\}$ in this case, denoted by BOR_{Mode} may be computed by replacing k with $(Y|x)_{Mode}$ in (23) and (24).

Summing up, suppose based on an information set available up to time point $(t-1)$, the one-step ahead to four-steps ahead forecasts of X and Y are $\mu_{x(t+j)}$ and $\mu_{y(t+j)}, j = 0,1,2,3$, respectively. The bivariate fan charts for X and Y may be plotted for each of the four time points to display the forecast coordinates and the joint distribution of the risk surrounding them. Then, suppose at time point t, the information on X is available as x_t. Using this, the forecasts $\mu_{x(t+1)}$, $\mu_{x(t+2)}$ and $\mu_{x(t+3)}$ are updated to $\mu''_{x(t+1)}$, $\mu''_{x(t+2)}$ and $\mu''_{x(t+3)}$. Then, the revised forecasts for Y would be $(Y_t|X_t = x_t)$, $E\{Y_{t+1}|X_{t+1} = \mu''_{x(t+1)}\}$, $E\{Y_{t+2}|X_{t+2} = \mu''_{x(t+2)}\}$ and $E\{Y_{t+3}|X_{t+3} = \mu''_{x(t+3)}\}$. Alternative revised forecasts can be considered as the respective modes. Accordingly, the conditional fan charts for all the four-time points may be constructed.

IV. Numerical Illustration

This section attempts to check the performance of the bivariate and conditional fan charts using inflation and real Gross Domestic Product (GDP) growth. The relation between inflation and real GDP growth is positive and linear up to a threshold and then the relationship changes to negative yet remaining linear (Mohanty *et.al.*, 2011). For this exercise, quarterly Consumer Price Index (CPI) and real Gross Domestic Product (GDP) in India were considered for the period from Q1:2004-05 to Q2:2019-20. The CPI index series prior to 2013 was obtained by back-casting the CPI-IW. One-quarter ahead to four-quarters ahead quarter-on-quarter inflation and quarter-on-quarter real GDP growth forecasts for each of the quarters from Q1:2016-17 to Q1:2018-19 were derived using a cointegration[5] framework (equation 27 and Table 1).

$$\Delta \log(CPI)_t = \alpha E_{t-1} + \beta_1 \Delta \log(GDP)_{t-1} + \beta_2 \Delta \log(GDP)_{t-2} + \gamma_1 \Delta \log(CPI)_{t-1} \\ + \gamma_2 \Delta \log(CPI)_{t-2} + \epsilon_t$$

$$\Delta \log(GDP)_t = \alpha' E_{t-1} + \beta'_1 \Delta \log(GDP)_{t-1} + \beta'_2 \Delta \log(GDP)_{t-2} + \gamma'_1 \Delta \log(CPI)_{t-1} \\ + \gamma'_2 \Delta \log(CPI)_{t-2} + \varphi_t$$

(27)

$(CPI)_t$ = t^{th} quarter CPI Index

$(GDP)_t$ = t^{th} quarter real GDP

E_{t-1} = $(t-1)^{th}$ quarter's error correction term

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[5] The two series *viz.*, log of seasonally adjusted CPI index and log of seasonally adjusted real GDP series are non-stationary at levels and stationary in first difference. P value for null hypothesis of no cointegrating equation is 0.042 and that for 1 cointegrating equation is 0.522, thus indicating presence of 1 cointegrating equation.

$$E_{t-1} = \log(GDP)_{t-1} - c - \delta \log(CPI)_{t-1}$$

ϵ_t	=	unexplained component
φ_t	=	unexplained component

Table 1: Results of Vector Error Correction Model

Variables	$\Delta \log(CPI)_t$	$\Delta \log(GDP)_t$
E_{t-1}	**−0.005**[*]	**−0.012**[*]
$\Delta \log(GDP)_{t-1}$	0.086	0.052
$\Delta \log(GDP)_{t-2}$	−0.016	−0.224
$\Delta \log(CPI)_{t-1}$	**0.356**[*]	−0.206
$\Delta \log(CPI)_{t-2}$	0.068	−0.062

Long term equation: $\log(GDP)_t = -14.154^* - 0.554 \log(CPI)_t$

Portmanteau test P value is 0.976 which indicates absence of autocorrelation in the residuals.
* significant at 5 per cent

The quarter-on-quarter inflation and growth forecasts were converted back into level series followed by transformation into year-on-year forecasts. The values of σ_x and σ_y were obtained on the basis of past 7 years' forecasts of X and Y. Also, to apply the assumptions in equation (8), the error series of forecasts of X and Y were found to satisfy test of normality[6]. Bivariate fan charts and conditional fan charts for one-quarter to four-quarters ahead inflation and growth forecasts were constructed for each of the forecast horizons separately for various combinations of p_x, p_y and θ, where, $tan(\theta) = m$. As, y-axis as a cut-line, distributes the probability symmetrically over the y-axis, thus imposing symmetric risks to the conditional forecasts of Y, hence, $\theta = 90°$ is not a preferable choice; however, $\theta = 89°$ has been studied. The performance of conditional fan charts for growth forecasts given available information on inflation (data on which is released about two months prior to the release of data on growth) were compared (Table 2) with the univariate fan charts for growth (for various combinations of p_x, p_y and θ) on the basis of:

i. Average absolute deviation (basis points) from actual growth

$$= \frac{1}{4} \sum_{i=1}^{4} |y_i - \mu_{yi}| * 100 \tag{28}$$

ii. Average absolute deviation (basis points) of actual growth from central path

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[6] P values for Jarque-Bera test applied on the error series of one-quarter ahead to four-quarters ahead inflation forecasts were 0.364, 0.779, 0.098 and 0.104, respectively. P values for Jarque-Bera test applied on the error series of one-quarter ahead to four-quarters ahead growth forecasts were 0.153, 0.135, 0.105 and 0.649, respectively.

$$= \frac{1}{4} \sum_{i=1}^{4} \left| x_i - \left(\frac{\mu_{yi}^{L} + \mu_{yi}^{U}}{2} \right) \right| * 100 \tag{29}$$

iii. Average width (basis points) of confidence band

$$= \frac{1}{4} \sum_{i=1}^{4} \left(\mu_{yi}^{U} - \mu_{yi}^{L} \right) * 100 \tag{30}$$

μ_{yi}^{L} =   lower confidence limit for μ_{yi}

μ_{yi}^{U} =   upper confidence limit for μ_{yi}

i =   forecast horizon

The combinations of p_x, p_y and θ for which the confidence intervals were the narrowest for the conditional fan charts were chosen as the pre-specifications for the bivariate fan charts shown in Annexure 2 and also the conditional fan charts displayed in Annexure 3.

It is observed from the bivariate fan charts in Annexure 2 that the spread of the fans increases with the increase in the forecast horizon. From one-quarter ahead to four-quarters ahead fans, movement of the centre of the fan towards right indicates that inflation is expected to rise, while movement towards left indicates the opposite. Similarly, centre of the fan moving upside indicates that growth is expected to increase and downward movement points to a possibility of slowdown in growth. While for few bivariate fans, the actual inflation and growth (x, y) coordinate lies at the centre of the fan, for others the performance of the fan chart is not so good, as it entirely depends on the forecasting model and past forecasts' performances.

The performance of conditional fans depends of the values of p_x and p_y, as it does not perform better than the univariate fan for all combinations of these parameters. Table 2 displays the performance of the conditional fans *vis-à-vis* the univariate fans for growth forecasts for those combinations of p_x, p_y and θ, where the conditional fans outperformed the univariate fans in all the three aspects given in (i), (ii) and (iii). While the confidence band for the conditional fan is narrower than the univariate fan in all these cases, the narrowest confidence bands are obtained when $\theta = 89°\,(m = 57.290)$. For such cases, mostly the conditional mean is a better revised forecast for growth than that of the conditional mode. The width of the confidence band depends on the values of σ_x and σ_y. Hence, better the performance of the past forecasts, narrower is the confidence band. In other words, the width of the confidence band is directly linked to the quality of the forecasting model used.

Table 2: Performance of Conditional Fan Charts

Quarter	θ	p_x	p_y	Average absolute deviation (basis points) from actual growth			Average absolute deviation (basis points) of actual growth from central path		Average Width (basis points) of confidence band		
				Univariate fan from Cointegration framework	Conditional fan from Bivariate Framework		Univariate fan from Cointegration framework	Conditional fan from Bivariate Framework	Band (per cent)	Univariate fan from Cointegration framework	Conditional fan from Bivariate Framework
					Mean	Mode					
Q1:2016-17 to Q4:2016-17	15°	0.5	0.5	81.8	81.0	81.0	81.8	81.0	50	481.5	470.0
									70	739.8	722.2
									90	1174.1	1146.2
	89°	0.7	0.3	81.8	48.8	80.5	187.1	57.1	50	647.7	421.7
									70	980.8	635.4
									90	1534.7	1008.2
	89°	0.9	0.2	81.8	111.5	81.0	487.3	104.3	50	969.4	454.6
									70	1454.4	676.6
									90	2237.6	1049.5
	95°	0.7	0.4	81.8	82.2	81.0	79.9	79.1	50	517.8	506.9
									70	792.5	773.6
									90	1253.2	1224.5
Q2:2016-17 to Q1:2017-18	0°	0.5	0.5	55.8	36.1	36.1	55.8	36.1	50	476.7	468.8
									70	732.6	720.3
									90	1162.6	1143.2
	89°	0.3	0.7	55.8	42.8	42.8	221.1	42.8	50	641.4	360.6
									70	971.2	554.2
									90	1519.6	879.5
	95°	0.7	0.3	55.8	42.9	42.8	305.8	42.8	50	641.4	360.7
									70	971.2	554.2
									90	1519.6	879.7
	89°	0.2	0.9	55.8	48.4	48.4	1423.7	48.4	50	1979.8	333.5
									70	2971.9	512.5
									90	4501.9	813.4
	95°	0.9	0.2	55.8	87.9	52.4	603.1	58.0	50	959.8	347.6
									70	1440.1	536.7
									90	2215.6	873.9
	89°	0.4	0.7	55.8	37.3	37.3	221.1	37.3	50	641.4	360.6
									70	971.2	554.2
									90	1519.6	879.5
	95°	0.7	0.4	55.8	41.4	41.4	151.9	41.4	50	512.7	398.4
									70	784.7	612.2
									90	1240.9	971.5
Q3:2016-17 to Q2:2017-18	45°	0.5	0.5	61.9	51.3	51.3	61.9	51.3	50	470.7	467.9
									70	723.3	718.9
									90	1147.9	1141.0
	89°	0.3	0.7	61.9	57.4	57.4	208.9	57.4	50	633.3	359.9
									70	958.9	553.1
									90	1500.4	877.7
	95°	0.7	0.3	61.9	56.7	57.4	311.4	57.2	50	633.3	360.2
									70	958.9	553.6
									90	1500.4	879.4
	89°	0.2	0.9	61.9	59.6	59.3	1396.3	59.3	50	1954.8	332.9
									70	2934.3	511.5

Period	Angle										
									90	4445.0	811.8
	95°	0.9	0.2	61.9	158.5	60.7	604.9	69.0	50	947.7	362.7
									70	1421.9	569.4
									90	2187.5	1475.7
	89°	0.4	0.7	61.9	55.5	55.5	208.9	55.5	50	633.3	359.9
									70	958.9	553.1
									90	1500.4	877.7
	95°	0.7	0.4	61.9	56.9	56.9	159.4	56.9	50	506.2	397.6
									70	774.8	611.0
									90	1225.2	969.6
Q4:2016-17 to Q3:2017-18	60°	0.5	0.5	70.7	67.9	67.9	70.7	67.9	50	463.4	462.2
									70	712.0	710.2
									90	1130.0	1127.2
	120°	0.3	0.7	70.7	349.5	70.3	326.8	324.5	50	623.4	621.3
									70	943.9	940.8
									90	1477.0	1472.2
	89°	0.7	0.3	70.7	60.8	69.5	185.3	67.6	50	623.4	357.5
									70	943.9	550.0
									90	1477.0	878.0
	0°	0.4	0.7	70.7	349.7	70.2	326.8	324.7	50	623.4	621.4
									70	943.9	941.0
									90	1477.0	1472.5
Q1:2017-18 to Q4:2017-18	75°	0.5	0.5	113.9	99.5	99.5	113.9	99.5	50	455.7	452.9
									70	700.3	696.0
									90	1111.4	1104.5
	89°	0.3	0.7	113.9	106.9	106.9	365.8	106.9	50	613.1	348.4
									70	928.4	535.4
									90	1452.7	849.7
	95°	0.7	0.3	113.9	97.7	107.0	137.9	104.9	50	613.1	351.0
									70	928.4	540.4
									90	1452.7	865.0
	89°	0.2	0.9	113.9	111.1	109.3	1515.4	109.4	50	1892.6	322.3
									70	2841.0	495.3
									90	4303.7	786.3
	89°	0.4	0.7	113.9	104.6	104.6	365.8	104.6	50	613.1	348.4
									70	928.4	535.4
									90	1452.7	849.7
Q2:2017-18 to Q1:2018-19	45°	0.7	0.3	177.9	49.9	187.2	69.9	29.4	50	603.3	575.1
									70	913.5	872.5
									90	1429.4	1371.3
	135°	0.2	0.9	177.9	1608.6	175.3	1556.9	1480.3	50	1862.3	1797.9
									70	2795.4	2699.9
									90	4234.6	4090.3
Q3:2017-18 to Q2:2018-19	30°	0.7	0.3	129.1	96.9	138.1	115.2	74.1	50	594.5	560.5
									70	900.2	849.9
									90	1408.6	1335.2
	75°	0.2	0.9	129.1	1511.7	125.0	1488.0	1394.3	50	1835.1	1722.3
									70	2754.7	2628.5
									90	4173.0	3960.7
	89°	0.9	0.2	129.1	120.8	130.9	390.7	129.7	50	889.7	312.2
									70	1334.9	480.2
									90	2053.7	765.4

Period	Angle								%		
Q4:2017-18 to Q3:2018-19	95°	0.5	0.5	101.0	63.0	63.0	101.0	63.0	50	436.1	411.0
									70	670.1	631.6
									90	1063.4	1002.3
	89°	0.3	0.7	101.0	84.4	84.4	341.9	84.4	50	586.7	316.2
									70	888.4	485.9
									90	1390.0	771.1
	89°	0.2	0.9	101.0	156.4	91.1	1442.0	95.0	50	1810.9	296.3
									70	2718.4	456.8
									90	4117.9	737.5
	75°	0.9	0.2	101.0	457.8	98.3	411.9	408.0	50	878.0	870.1
									70	1317.3	1303.7
									90	2026.6	1974.0
	89°	0.4	0.7	101.0	77.7	77.7	341.9	77.7	50	586.7	316.2
									70	888.4	485.9
									90	1390.0	771.1
Q1:2018-19 to Q4:2018-19	89°	0.3	0.7	14.3	11.9	11.9	252.7	11.9	50	580.5	301.3
									70	879.0	463.0
									90	1375.4	734.8
	95°	0.7	0.3	14.3	47.9	11.7	224.2	21.0	50	580.5	310.8
									70	879.0	481.9
									90	1375.4	811.1
	89°	0.2	0.9	14.3	6.1	8.1	1341.2	7.9	50	1791.9	278.9
									70	2689.9	428.6
									90	4074.7	680.8
	85°	0.4	0.7	14.3	11.9	22.4	252.7	19.8	50	580.5	303.9
									70	879.0	468.0
									90	1375.4	750.5
Q2:2018-19 to Q1:2019-20	85°	0.3	0.7	30.5	12.7	50.0	205.3	34.8	50	573.9	312.1
									70	869.1	487.6
									90	1359.9	980.0
	85°	0.4	0.7	30.5	12.6	57.8	205.3	40.9	50	573.9	314.0
									70	869.1	491.9
									90	1359.9	999.7
Q3:2018-19 to Q1:2019-20	85°	0.3	0.7	53.3	21.5	65.6	183.5	45.5	50	566.5	321.5
									70	857.9	506.9
									90	1342.3	1078.8
	85°	0.4	0.7	53.3	20.9	72.2	183.5	48.7	50	566.5	323.6
									70	857.9	511.7
									90	1342.3	1258.4

V. Conclusion

The paper proposes the construction of bivariate fan chart using joint distribution of two related variables with their forecasts as the modal coordinate of the distribution and also incorporating asymmetrical risks to the forecasts. Further, the conditional fan chart of one variable given the known information on the other is also derived and the conditional mean is proposed as the revised forecast for the variable. Given suitable values of the balance of risks of the two variables, the conditional fan chart performs better than the univariate fan charts in terms of absolute deviation of the realised values from the forecasts, absolute deviation of the

realised values from the central path of the confidence band and width of the confidence band. The proposed bivariate and conditional fan charts are appropriate in a scenario in which information on the two target variables are released at different lags because additional information on one variable refines the forecast of the other for which information will be released later. Although, the relationship between two related variables can be taken care of in the forecasting model itself, such relationship should also be visible in the error distribution surrounding their forecasts, thus proving the utility of the bivariate fan chart. The information on effects of risk to forecast of one variable on the risk to forecast of the other is important from the policy perspective.

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Marginal distribution of X

$$g(x) = \int_{-\infty}^{\infty} f(x,y)dy$$

$$= C \int_{-\infty}^{\mu_y+m(x-\mu_x)} exp\left[-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_x}{\sigma_{1x}}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_{1x}}\right)\left(\frac{y-\mu_y}{\sigma_{1y}}\right) + \left(\frac{y-\mu_y}{\sigma_{1y}}\right)^2\right\}\right]dy$$

$$+ C \int_{\mu_y+m(x-\mu_x)}^{\infty} exp\left[-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_x}{\sigma_{2x}}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_{2x}}\right)\left(\frac{y-\mu_y}{\sigma_{2y}}\right) + \left(\frac{y-\mu_y}{\sigma_{2y}}\right)^2\right\}\right]dy$$

$$= C\sigma_{1y}\sqrt{1-\rho^2}exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_{1x}^2}\right\} \int_{-\infty}^{\frac{\frac{m(x-\mu_x)}{\sigma_{1y}} - \frac{\rho(x-\mu_x)}{\sigma_{1x}}}{\sqrt{1-\rho^2}}} exp\left(-\frac{u^2}{2}\right)du$$

$$+ C\sigma_{2y}\sqrt{1-\rho^2}exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_{2x}^2}\right\} \int_{\frac{\frac{m(x-\mu_x)}{\sigma_{2y}} - \frac{\rho(x-\mu_x)}{\sigma_{2x}}}{\sqrt{1-\rho^2}}}^{\infty} exp\left(-\frac{u^2}{2}\right)du$$

$$= C\sigma_{1y}\sqrt{1-\rho^2}\sqrt{2\pi}\,exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_{1x}^2}\right\}\Phi\left\{\frac{\frac{m(x-\mu_x)}{\sigma_{1y}} - \frac{\rho(x-\mu_x)}{\sigma_{1x}}}{\sqrt{1-\rho^2}}\right\}$$

$$+ C\sigma_{2y}\sqrt{1-\rho^2}\sqrt{2\pi}\,exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_{2x}^2}\right\}\Phi\left\{-\frac{\frac{m(x-\mu_x)}{\sigma_{2y}} - \frac{\rho(x-\mu_x)}{\sigma_{2x}}}{\sqrt{1-\rho^2}}\right\}, \quad x \in \mathbb{R}$$

Conditional Expectation $E(Y|x)$

$$E(Y|x) = \int_{-\infty}^{\infty} yf_Y(y|x)dy$$

$$= \frac{C}{g(x)} \int_{-\infty}^{\mu_y+m(x-\mu_x)} yexp\left[-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_x}{\sigma_{1x}}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_{1x}}\right)\left(\frac{y-\mu_y}{\sigma_{1y}}\right) + \left(\frac{y-\mu_y}{\sigma_{1y}}\right)^2\right\}\right]dy$$

$$+ \frac{C}{g(x)} \int_{\mu_y+m(x-\mu_x)}^{\infty} yexp\left[-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_x}{\sigma_{2x}}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_{2x}}\right)\left(\frac{y-\mu_y}{\sigma_{2y}}\right) + \left(\frac{y-\mu_y}{\sigma_{2y}}\right)^2\right\}\right]dy$$

$$= \frac{C\sigma_{1y}^2\sqrt{1-\rho^2}}{g(x)} exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_{1x}^2}\right\}\left\{\sqrt{1-\rho^2}\int\limits_{-\infty}^{\frac{m(x-\mu_x)}{\sigma_{1y}}-\frac{\rho(x-\mu_x)}{\sigma_{1x}}}{\sqrt{1-\rho^2}}}\ u\ exp\left(-\frac{u^2}{2}\right)du\right.$$

$$\left.+\frac{\rho(x-\mu_x)}{\sigma_{1x}}\int\limits_{-\infty}^{\frac{m(x-\mu_x)}{\sigma_{1y}}-\frac{\rho(x-\mu_x)}{\sigma_{1x}}}{\sqrt{1-\rho^2}}}\ exp\left(-\frac{u^2}{2}\right)du\right\}$$

$$+\frac{C\sigma_{1y}\mu_y\sqrt{1-\rho^2}}{g(x)} exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_{1x}^2}\right\}\int\limits_{-\infty}^{\frac{m(x-\mu_x)}{\sigma_{1y}}-\frac{\rho(x-\mu_x)}{\sigma_{1x}}}{\sqrt{1-\rho^2}}}\ exp\left(-\frac{u^2}{2}\right)dw$$

$$+\frac{C\sigma_{2y}^2\sqrt{1-\rho^2}}{g(x)} exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_{2x}^2}\right\}\left\{\sqrt{1-\rho^2}\int\limits_{\frac{m(x-\mu_x)}{\sigma_{2y}}-\frac{\rho(x-\mu_x)}{\sigma_{2x}}}{\sqrt{1-\rho^2}}}^{\infty}\ u\ exp\left(-\frac{u^2}{2}\right)du\right.$$

$$\left.+\frac{\rho(x-\mu_x)}{\sigma_{2x}}\int\limits_{\frac{m(x-\mu_x)}{\sigma_{2y}}-\frac{\rho(x-\mu_x)}{\sigma_{2x}}}{\sqrt{1-\rho^2}}}^{\infty}\ exp\left(-\frac{u^2}{2}\right)du\right\}$$

$$+\frac{C\sigma_{2y}\mu_y\sqrt{1-\rho^2}}{g(x)} exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_{2x}^2}\right\}\int\limits_{\frac{m(x-\mu_x)}{\sigma_{2y}}-\frac{\rho(x-\mu_x)}{\sigma_{2x}}}{\sqrt{1-\rho^2}}}^{\infty}\ exp\left(-\frac{u^2}{2}\right)dw$$

$$=\frac{C\sqrt{1-\rho^2}}{g(x)}\sigma_{1y}^2 exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_{1x}^2}\right\}\left[-\sqrt{2\pi}\sqrt{1-\rho^2}\ \phi\left\{\frac{\frac{m(x-\mu_x)}{\sigma_{1y}}-\frac{\rho(x-\mu_x)}{\sigma_{1x}}}{\sqrt{1-\rho^2}}\right\}\right.$$

$$\left.+\frac{\rho(x-\mu_x)}{\sigma_{1x}}\sqrt{2\pi}\Phi\left\{\frac{\frac{m(x-\mu_x)}{\sigma_{1y}}-\frac{\rho(x-\mu_x)}{\sigma_{1x}}}{\sqrt{1-\rho^2}}\right\}\right]$$

$$+\frac{C\sqrt{1-\rho^2}}{g(x)}\sigma_{1y}\mu_y\sqrt{2\pi}exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_{1x}^2}\right\}\Phi\left\{\frac{\frac{m(x-\mu_x)}{\sigma_{1y}}-\frac{\rho(x-\mu_x)}{\sigma_{1x}}}{\sqrt{1-\rho^2}}\right\}$$

$$+\frac{C\sqrt{1-\rho^2}}{g(x)}\sigma_{2y}{}^2exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_{2x}^2}\right\}\left[\sqrt{2\pi}\sqrt{1-\rho^2}\,\phi\left\{\frac{\frac{m(x-\mu_x)}{\sigma_{2y}}-\frac{\rho(x-\mu_x)}{\sigma_{2x}}}{\sqrt{1-\rho^2}}\right\}\right.$$

$$\left.+\frac{\rho(x-\mu_x)}{\sigma_{2x}}\sqrt{2\pi}\Phi\left\{-\frac{\frac{m(x-\mu_x)}{\sigma_{2y}}-\frac{\rho(x-\mu_x)}{\sigma_{2x}}}{\sqrt{1-\rho^2}}\right\}\right]$$

$$+\frac{C\sqrt{1-\rho^2}}{g(x)}\sigma_{2y}\mu_y\sqrt{2\pi}exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_{2x}^2}\right\}\Phi\left\{-\frac{\frac{m(x-\mu_x)}{\sigma_{2y}}-\frac{\rho(x-\mu_x)}{\sigma_{2x}}}{\sqrt{1-\rho^2}}\right\}$$

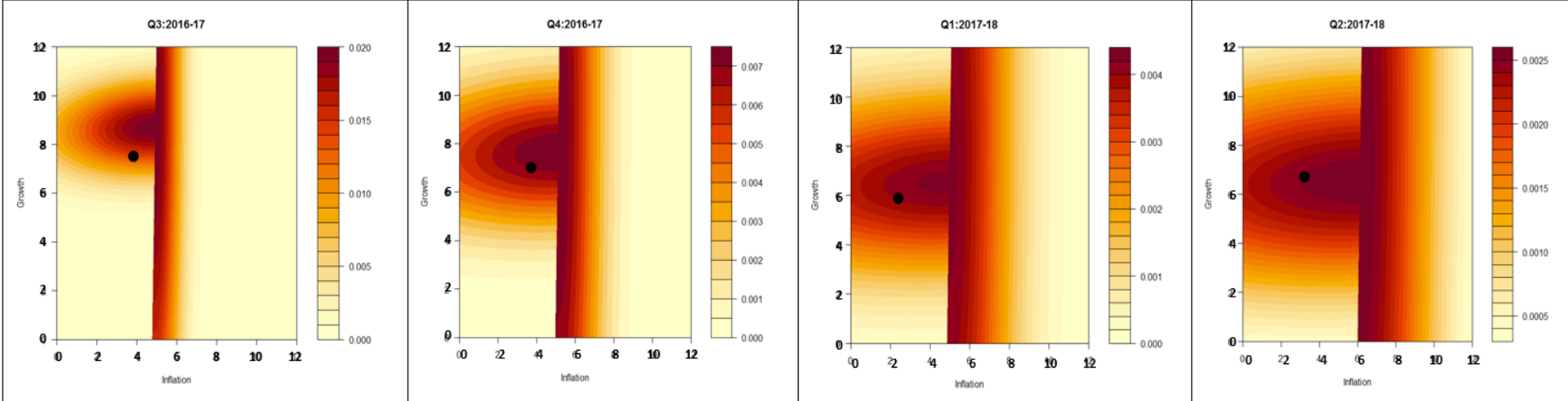# Annexure 2 - Bivariate Fan Charts for Inflation and Growth
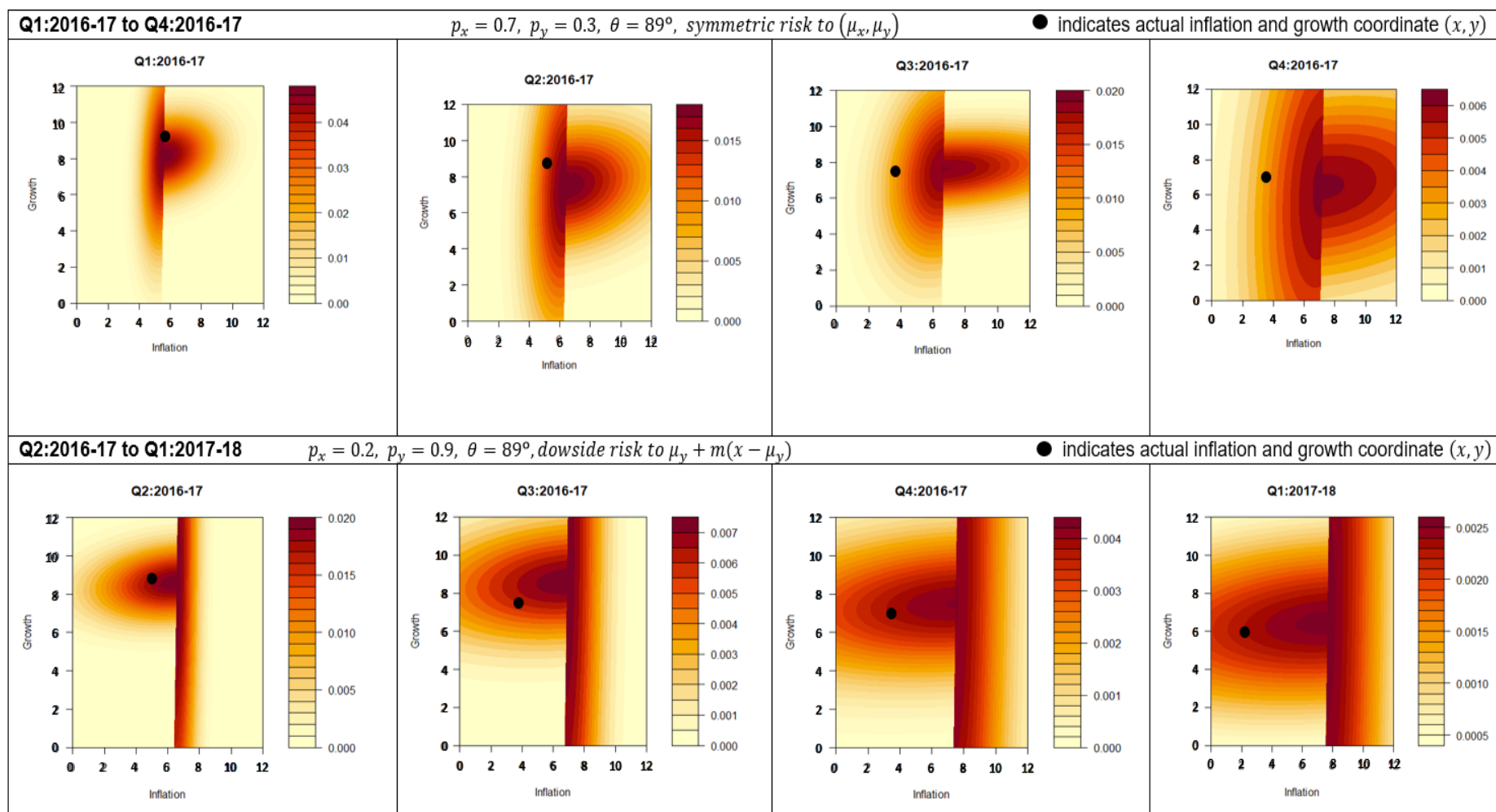
Q1:2016-17 to Q4:2016-17 — $p_x = 0.7,\ p_y = 0.3,\ \theta = 89°,\ symmetric\ risk\ to\ (\mu_x, \mu_y)$ — ● indicates actual inflation and growth coordinate (x, y)

Q2:2016-17 to Q1:2017-18 — $p_x = 0.2,\ p_y = 0.9,\ \theta = 89°,\ dowside\ risk\ to\ \mu_y + m(x - \mu_y)$ — ● indicates actual inflation and growth coordinate (x, y)

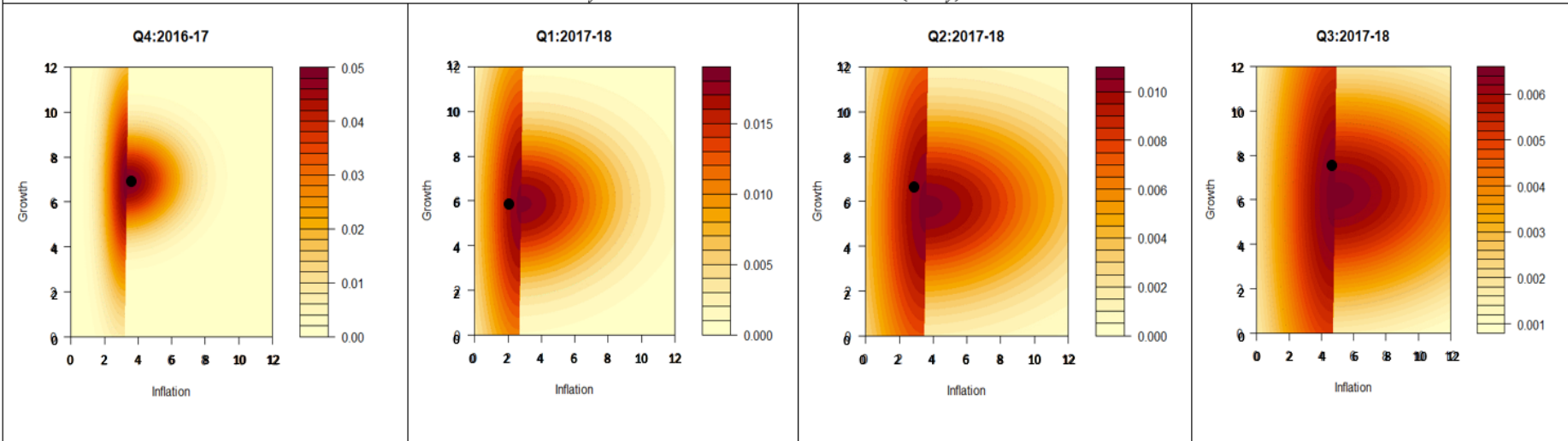**Q3:2016-17 to Q2:2017-18** $p_x = 0.2,\ p_y = 0.9,\ \theta = 89°,\ dowside\ risk\ to\ \mu_y + m(x - \mu_y)$ ● indicates actual inflation and growth coordinate (x, y)
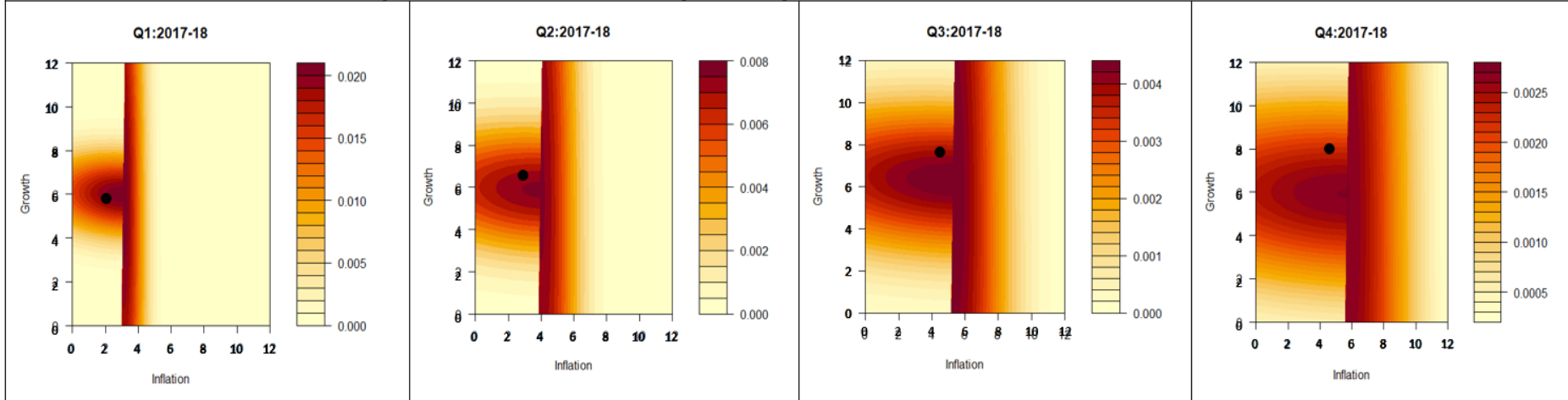
**Q4:2016-17 to Q3:2017-18** $p_x = 0.7,\ p_y = 0.3,\ \theta = 89°,\ symmetric\ risk\ to\ (\mu_x, \mu_y)$ ● indicates actual inflation and growth coordinate (x, y)
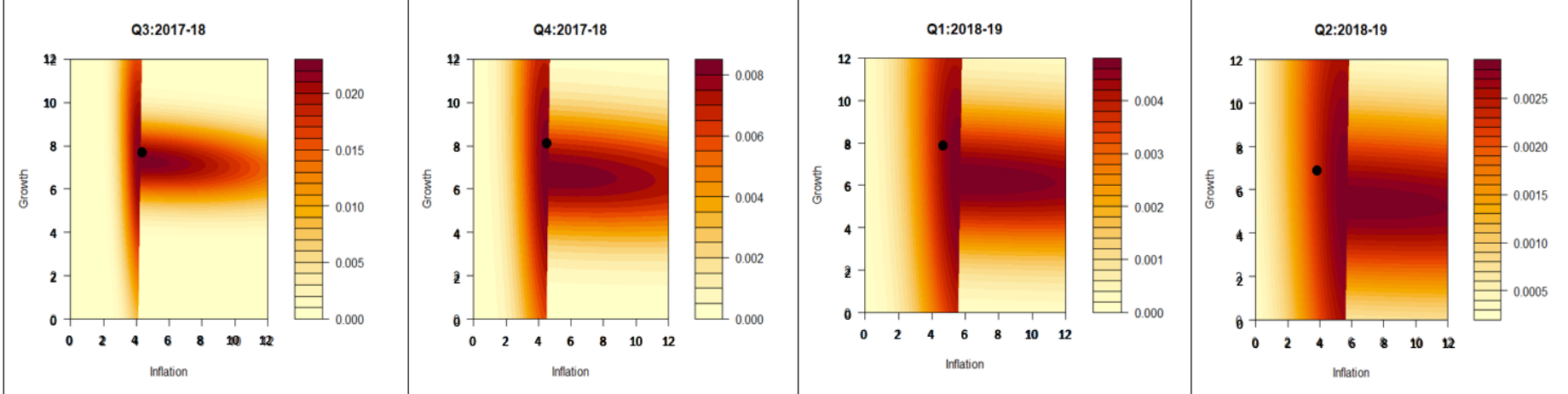
27

Q1:2017-18 to Q4:2017-18 $p_x = 0.2, \; p_y = 0.9, \; \theta = 89°, \; dowside \; risk \; to \; \mu_y + m(x - \mu_y)$ ● indicates actual inflation and growth coordinate (x, y)

Q1:2017-18   Q2:2017-18   Q3:2017-18   Q4:2017-18

Q2:2017-18 to Q1:2018-19 $p_x = 0.7, \; p_y = 0.3, \; \theta = 45°, downside \; risk \; to \; \mu_x \; combined \; with \; upside \; risk \; to \; \mu_y + m(x - \mu_x)$ ● indicates actual inflation and growth coordinate (x, y)

Q2:2017-18   Q3:2017-18   Q4:2017-18   Q1:2018-19

28
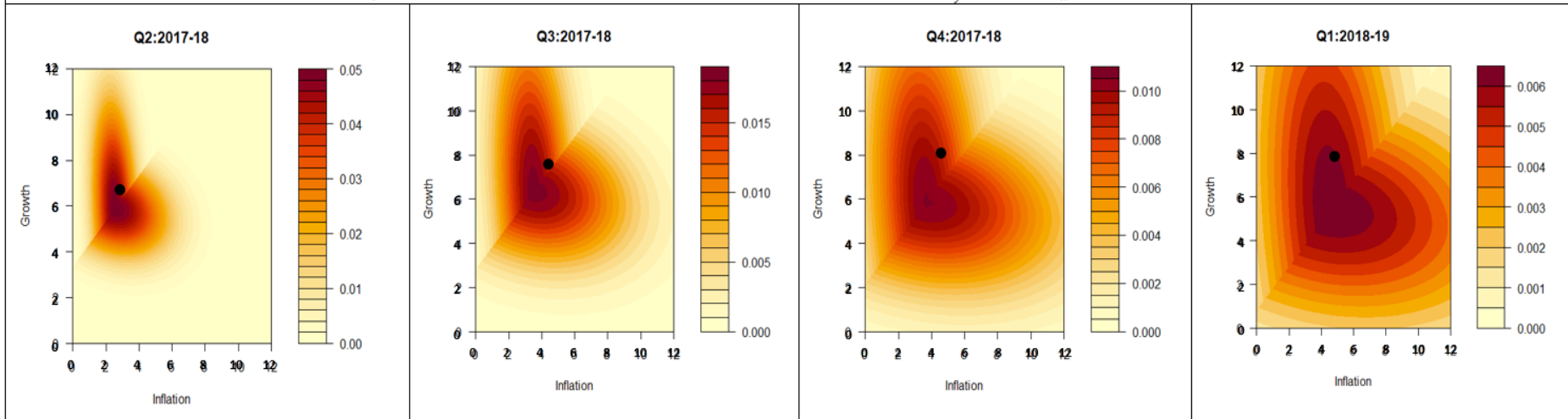
Q3:2017-18 to Q2:2018-19 $p_x = 0.9, \ p_y = 0.2, \ \theta = 89°, \ dowside \ risk \ to \ \mu_y + m(x - \mu_y)$ ● indicates actual inflation and growth coordinate (x, y)

Q4:2017-18 to Q3:2018-19 $p_x = 0.4, \ p_y = 0.7, \ \theta = 89°, \ dowside \ risk \ to \ \mu_y + m(x - \mu_y)$ ● indicates actual inflation and growth coordinate (x, y)
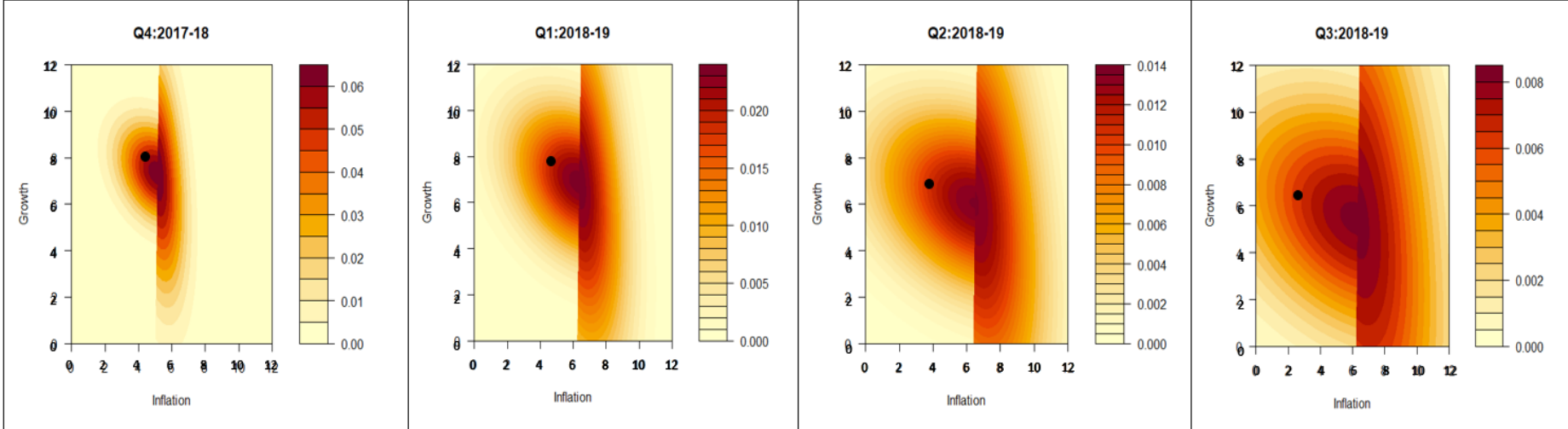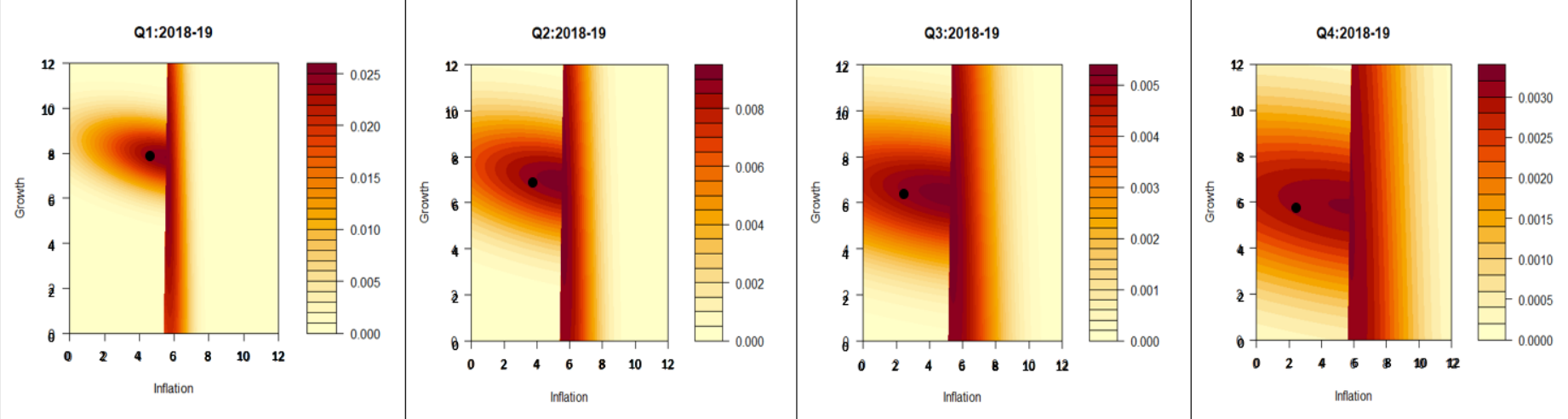
Q1:2018-19 to Q4:2018-19 $p_x = 0.2$, $p_y = 0.9$, $\theta = 89°$, *upside risk to* μ_x *combined with downside risk to* $\mu_y + m(x - \mu_x)$ ● indicates actual inflation and growth coordinate (x, y)

Q2:2018-19 to Q1:2019-20 $p_x = 0.3$, $p_y = 0.7$, $\theta = 85°$, *upside risk to* μ_x *combined with downside risk to* $\mu_y + m(x - \mu_x)$ ● indicates actual inflation and growth coordinate (x, y)

Q3:2018-19 to Q2:2019-20 $p_x = 0.3$, $p_y = 0.7$, $\theta = 85°$, *upside risk to* μ_x *combined with downside risk to* $\mu_y + m(x - \mu_x)$ ● indicates actual inflation and growth coordinate (x, y)

Q3:2018-19   Q4:2018-19   Q1:2019-20   Q2:2019-20

Annexure 3 - Conditional Fan Charts for Growth

Q1:2016-17 to Q4:2016-17
$p_x = 0.7,$ $p_y = 0.3,$ $\theta = 89°,$ *upside bias*

Q2:2016-17 to Q1:2017-18
$p_x = 0.2,$ $p_y = 0.9,$ $\theta = 89°,$ *symmetric bias*

Q3:2016-17 to Q2:2017-18
$p_x = 0.2,$ $p_y = 0.9,$ $\theta = 89°,$ *symmetric bias*

Q4:2016-17 to Q3:2017-18
$p_x = 0.7,$ $p_y = 0.3,$ $\theta = 89°,$ *symmetric bias*

Q1:2017-18 to Q4:2017-18
$p_x = 0.2,$ $p_y = 0.9,$ $\theta = 89°,$ *symmetric bias*

Q2:2017-18 to Q1:2018-19
$p_x = 0.7,$ $p_y = 0.3,$ $\theta = 45°,$ *upside bias*

Q3:2017-18 to Q2:2018-19
$p_x = 0.9,$ $p_y = 0.2,$ $\theta = 89°,$ *symmetric bias*

Q4:2017-18 to Q3:2018-19
$p_x = 0.4,$ $p_y = 0.7,$ $\theta = 89°,$ *symmetric bias*

Q1:2018-19 to Q4:2018-19

$p_x = 0.2$, $p_y = 0.9$, $\theta = 89°$, *symmetric bias*

Q2:2018-19 to Q1:2019-20

$p_x = 0.3$, $p_y = 0.7$, $\theta = 85°$, *symmetric bias*

Q3:2018-19 to Q2:2019-20

$p_x = 0.3$, $p_y = 0.7$, $\theta = 85°$, *downside bias*