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A New Unit Root Test Criterion

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In this paper, a new criterion is proposed to test the presence of a unit root in any zero-mean time series data with no deterministic trend and no structural break. The test is developed based on the ratio of the Probability Density Functions (PDFs) of the data under the null of presence of a unit root to the alternative of stationarity. As the distribution of the test statistic is non-standard, the Monte Carlo simulation (MCS) technique has been used to determine the empirical probability distribution of the test statistic. MCS is also used to compare the power of the test for a finite sample with select univariate unit-root tests that are commonly used in empirical research, namely the ADF test, Phillips-Perron test, KPSS test, ERS test, Zivot and Andrews test, Schmidt and Phillips test, Pantula, Gonzales-Farias and Fuller test, and Breitung's variance ratio test. The paper demonstrates higher power of the new test *vis-à-vis* the existing tests for sample sizes under 50. For large sample sizes, its power is either higher or on-par with the other tests.

JEL Classification: C22, C53, B23

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Introduction

Any time series data *i.e.*, a sequence of data points arranged to reflect its evolution over time is an integral part of economic analysis, testing of various economic hypotheses, and statistical modelling for forecasting. However, if the time series data are not stationary, then the inferences derived from the analysis can be misleading. If a time series is not stationary, but its first difference is stationary, then the data generating process is called the unit root process.

There are many standard methods for testing of unit roots in the literature. The empirical power of these unit root tests, however, is found to be low especially for small samples. The power of a test signifies how well the test can correctly identify a time series as stationary when the series is indeed stationary. This paper proposes a new criterion to test the unit root hypothesis and compares its power with various available unit root tests.

The paper is organised as follows: Section II contains a review of the literature, while Section III lays out the methodology for the new test statistic. The test outcome is compared with the other existing tests in Section IV, followed by conclusions in Section V.

Section II
Literature Review

An autoregressive process of order p, *i.e.,* AR(p) is defined as under,

$$x_t = \alpha + \beta_1 x_{t-1} + \beta_2 x_{t-2} + \cdots + \beta_p x_{t-p} + \varepsilon_t \qquad ...(1)$$

$$=> \beta(L) x_t = \alpha + \varepsilon_t \qquad ...(2)$$

where, α is a constant; β_1, β_2,...β_p are coefficients; and ε_t is independent and identically distributed (iid) over time and follows N(0,σ^2); $\beta(L)$ is a lag-polynomial, such that, $\beta(L) = 1 - \beta_1 L - \beta_2 L^2 - \cdots - \beta_p L^p$; L is the lag operator, such that, for some m, $L^m x_t = x_{t-m}$.

The process (1) is strictly stationary if for any $q_1, q_2, \cdots q_n$, the joint PDF of $(x_{t+q_1}, x_{t+q_2}, \cdots . x_{t+q_n})$ depends only on lag lengths ($q_1, q_2, \cdots q_n$) and not on the time 't'. If $E(x_t) = E(x_{t-q})$ and covariance

$E(x_t - E(x_t))(x_{t-q} - E(x_{t-q}))$ only depends on the lag length 'q' and not on time 't' then the series is said to be weak or covariance stationary. Alternatively, if all the roots of $A(y) = 1 - \beta_1 y - \beta_2 y^2 - \cdots - \beta_p y^p = 0$ lie outside the unit circle, then y_i is weakly stationary. On the other hand, the process is non-stationary when the roots lie inside the unit circle, making it an explosive series. When the roots lie on the unit circle then the process is said to have at least one unit root and the number of unit roots determine the order of integration. If a process is integrated of order k *i.e.,* I(k) then k is the minimum number of differences required to make the process stationary.

The fixed parameter AR (1) process is written as follows:

$$x_t = \alpha + \beta\, x_{t-1} + \varepsilon_t \qquad ...(3)$$

Equation (3) is stationary for $\beta < 1$. ε_t is the disturbance term which has only a transitory effect on x_t; autocorrelation coefficients of order k, *i.e.,* ρ_k diminish with increase of lag k, and the sum of ρ_k is finite. If $\beta=1$ then the AR (1) process in equation (3) has a unit root and is termed as random walk series with drift α, and the accumulated random component $\sum_{t=1}^{T} \varepsilon_t$ will produce a stochastic trend and will have a permanent effect on x_t. Further, when $\beta=1$, the series (3) becomes a random walk series with a drift, and it will have both deterministic trend ($\alpha * t$) as well as stochastic trend ($\sum_{t=1}^{T} \varepsilon_t$) and both contribute to the non-stationary characteristics of x_t (Solberger, 2013). Random walk with a drift can be written as:

$$x_t = \alpha + x_{t-1} + \varepsilon_i = \alpha + (\alpha + x_{t-2} + \varepsilon_{t-1}) + \varepsilon_t$$

$$x_T = x_0 + \alpha T + \sum_{t=1}^{T} \varepsilon_t \qquad ...(4)$$

There are many tests for the unit root hypothesis testing in autoregressive processes. The commonly used ones are Dickey and Fuller's ADF test (Dickey and Fuller, 1979); Phillips-Perron test (Phillips and Perron, 1988); KPSS (Kwiatkowski, Phillips, Schmidt and Shin, 1992) test; Elliot, Rothenberg, and Stock (ERS, 1996) test; Zivot and Andrews (ZA, 1992) test; Schmidt and Phillips (SP) test; Pantula, Gonzales-Farias and Fuller (PGFF, 1994) test; and Breitung variance ratio (BVR, 2002) test.

To test if a time series is nonstationary, the standard unit-root tests employ model (3) and consider the null hypothesis (H_0) and the alternative hypothesis (H_1) as follows:

H_0: $\beta = 1$

H_1: $\beta < 1$

These tests use mainly least-squares estimate (LSE) of β and the test statistic is the t-ratio of the estimate of β and its standard error.

The Dickey-Fuller (DF) (Dickey and Fuller, 1979) unit-root test is based on the model of the first-order autoregressive process as in (3). To formulate the test statistic, in equation (3), x_{t-1} is subtracted from both the sides:

$$\Delta x_t = \varphi x_{t-1} + \varepsilon_t;\ where, \varphi = \beta - 1 \qquad ...(5)$$

DF statistic is defined as: $t_{DF} = \frac{\hat{\beta} - 1}{s_{\hat{\beta}}}$ ...(6)

where, $\hat{\beta}$ is the least square estimate of β, and $S_{\hat{\beta}}$ is the standard error estimate. Under H0, t_{DF} follows the Dickey-Fuller distribution and the critical values are obtained through simulation and are tabulated in Dickey-Fuller (1979). If the computed value of t_{DF} exceeds a critical value at a chosen significance level, then the null hypothesis about the presence of a unit-root in a time series cannot be rejected.

Since the right-hand side of equation (5) contains lagged x_t *i.e.*, x_{t-1}, the disturbance terms ε_t are correlated. To take care of the autocorrelation, augmented Dickey-Fuller (ADF) test includes the lagged values of differences of x_t in the right-hand side of (5); it also includes a constant term c_t, which can be a pure constant or a linear time trend.

To verify the presence of unit-root in an AR(p), model (3) is extended to

$$\Delta x_t = c_t + \beta x_{t-1} + \sum_{j=1}^{p-1} \varphi_t \Delta x_{t-j} + \varepsilon_t \qquad ...(7)$$

ADF test statistic is: $t_{ADF} = \frac{\hat{\beta} - 1}{s_{\hat{\beta}}}$ ...(8)

where, $\hat{\beta}$ is the least square estimate of β in (6) and $S_{\hat{\beta}}$ is the standard error estimate of $\hat{\beta}$. Critical values for t_{ADF} under H_0 are tabulated in MacKinnon (1991).

The Phillips-Perron (PP) unit root test builds on the ADF test but it differs from the ADF test mainly in how it deals with the serial autocorrelation and heteroskedasticity in the errors. The null hypothesis in the PP test assumes that the process has a unit root, and the test statistics are given as follows (Pesaran, 2015):

$$Z_{\beta} = T(\hat{\beta} - 1) - \frac{1}{2}\frac{T^2 S_{\beta}{}^2}{S_T^2}(S_{LT}^2 - S_T^2) \quad \text{...(9)}$$

$$Z_{\mathrm{T}} = \left(\frac{S_T}{S_{LT}}\right) T_{DF} - \frac{1}{2}(S_{LT}^2 - S_T^2)\ \frac{1}{S_{LT}}\frac{T.S_{\hat{\beta}}}{S_T} \quad \text{...(10)}$$

where, $T_{\mathrm{DF}} = \frac{(\hat{\beta}-1)}{S_{\hat{\beta}}}$; $S_T^2 = \frac{1}{2}\sum_{i=1}^{T} \hat{\varepsilon}_i^2$; $S_{LT}^2 = S_T^2 + 2\sum_{j=1}^{q}\left(1 - \frac{j}{q+1}\right)\hat{\gamma}_{j,T}$;

and $\hat{\gamma}_{j,T} = \frac{1}{T}\sum_{i=j+1}^{T} \hat{\varepsilon}_i \hat{\varepsilon}_{i-j}$

If ε_i is i.i.d., then it implies $\hat{\gamma}_{j,T} = 0$ and $S_{LT}^2 = S_T^2$, and the limiting distribution of the test statistics reduces to DF test statistics.

Unlike ADF test, the KPSS test has a null of stationarity of a series including deterministic trend, and the alternative hypothesis is that the series is nonstationary due to the presence of a unit root. According to the KPSS test, time series observations x_t are decomposed as the sum of the deterministic trend, a random walk, and a stationary error term.

$$x_t = \eta.t + r_t + \varepsilon_t \quad \text{...(11)}$$

$$r_t = r_{t-1} + u_t \quad \text{....(12)}$$

where, t is the deterministic trend, r_t is the random walk process, ε_t is the stationary error, and u_t is i.i.d. error term with zero mean and constant variation σ_u^2. Under the null hypothesis of stationarity, σ_u^2=0 and if $\eta \neq 0$ then x_t is trend stationary with a deterministic trend. When σ_u^2>0 then x_t is non-stationary with a stochastic trend. The KPSS test introduces one-side Lagrange Multiplier (LM) test of null hypothesis of stationarity *i.e.,* σ_u^2=0 with assumption that u_t follows a normal distribution and $\varepsilon_t \sim$ i.i.d. N (0, σ_ε^2).

Instead of an LSE of β in equation (1), many studies employ the maximum likelihood estimate (MLE) and observe that the test statistic associated with the exact MLE, under alternative hypothesis of stationarity is more powerful than the LSE used in DF test (Pantula *et al.*, 1994; Fuller, 1996).

Elliott, Rothenberg, and Stock (1996) developed an asymptotically efficient test based on likelihood ratio assuming the data generating process as auto regressive (AR) of order 1 model with fixed parameter and obtained Gaussian power envelops for unit root tests. Jansson and Nielson (2012) studied the large sample property of a quasi-likelihood ratio test based on a Gaussian likelihood and showed that this test is nearly efficient. Jansson and Nielsen (2012) used the zero-mean Gaussian AR(1) model, in which $\{y_t$: 1 < t < T} was generated as $y_t = \rho y_{t-1} + \varepsilon_t$; where $y_0 = 0$ and $\varepsilon_t \sim$ i.i.d. N(0,1). The likelihood ratio test associated with the unit root testing problem H_0: ρ=1 versus H_0: ρ<1 was rejected for large values of $LR_T = max_{\rho \leq 1} L_T(\rho) - L_T(1)$; where $L_T(\rho) = -\frac{1}{2}\sum_{t=1}^{T}(y_t - \rho y_{t-1})^2$ is, up to a constant, the log likelihood function. LR_ρ was maximised for $\rho \leq 1$ *i.e.,* over H_o U H_1.

Skrobotov (2018) investigated the bootstrap implementation of the test as proposed by Jansson and Nielson (2012) and observed that likelihood ratio test produced poor finite sample properties when errors were strongly autocorrelated and noted that as compared to bootstrap ADF, the bootstrap likelihood ratio test exhibited better finite sample properties in certain cases.

Section III
Methodology

In this paper, we consider a zero-mean first order autoregressive process *i.e.,* AR (1) with time-varying coefficients as a stationary series as defined in (13), and a nonstationary series (*i.e.,* random walk series) with only stochastic trend component as defined in (14), and develop a test statistic to identify whether a given series is non-stationary (H_0) or stationary (H_1).

$$x_t = \beta_t x_{t-1} + \varepsilon_t \quad ...(13)$$

$$x_t = x_{t-1} + \varepsilon_t \quad ...(14)$$

where, t = 1, 2,…,T; $\beta_t < 1$ is the time-varying parameter associated with the 't'th observation; $\varepsilon_t \sim$ i.i.d. N(0,σ^2); and we assume $x_0 = 0$.

AR (1) model is used here with time-varying coefficient (β_t) rather than fixed coefficient β to avoid the strong assumptions made by many other studies that $x_t - \beta * x_{t-1}$ are independent and identically distributed (i.i.d.)

with a known distribution, which seems practically implausible. Instead, $x_t - \beta_t * x_{t-1}$ may be more likely i.i.d. in many real applications.

The test criterion has been developed assuming the time-varying coefficient (β_t) of AR (1) model to make it generic. The test criterion can be applied to a time series irrespective of practitioner's assumption on time variant coefficients or orders of AR/ MA process.

The test criterion is developed assuming that if the observed data series $'x'$ is indeed generated out of a random walk process, then it would look relatively more probable (or higher likelihood) when fitting it using the PDF of a unit root process rather than force fitting it with the PDF of a stationary process. The critical values or the rejection region of the test statistic is obtained using Monte Carlo Simulation (MCS) method.

Further, this paper empirically tests the efficiency of the proposed test in terms of its power, and the proportion of correctly identified series, as compared to other commonly used tests, such as ADF test, Phillips-Perron test, KPSS test, ERS test, Zivot and Andrews (ZA) test, Schmidt and Phillips (SP) test, Pantula, Gonzales-Farias and Fuller (PGFF) test, and Breitung's variance ratio (BVR) test on a set of simulated stationary (fixed and time-varying AR(1) and AR(2) models) and nonstationary data.

Given a time series observation $x = (x_1, x_2, \dots, x_T)$, we need to ascertain whether it is generated from a random walk process (null hypothesis: H_o) or a stationary process (Alternative hypothesis: H_1). Here, $(x_1, x_2, \dots, x_T)$ are not a mere multivariate sample but a time ordered data from a family of random variables. The time series characteristic is embedded in the construction of $(x_1, x_2, \dots, x_T)$ and their dependence structure is captured in variance-covariance matrix. The AR (1) stationary series with time-varying parameter as defined in (13) is:

$x_t = \beta_t x_{t-1} + \varepsilon_t$; where β_t<1 for t=1(1)T

where, $x_0 = 0 \; and \; \varepsilon_t \sim \text{i.i.d.}\, \text{N}(0, \sigma^2)$.

The null hypothesis, H_0: β_t=1 and Alternative hypothesis: H_1: 0< β_t<1 $\forall \; \text{t} = 1(1)\text{T}$.

III.1. PDF of a Random Walk Process (β_t=1 in equation (13))

$$x_1 = e_1;\ x_2 = x_1 + e_2 = e_1 + e_2;\ x_3 = x_2 + e_3 = e_1 + e_2 + e_3;\ \ldots;\ x_T = e_1 + e_2 + .. + e_T;$$

Therefore, $V(x_1) = \sigma^2$; $V(x_2) = 2\sigma^2$, $V(x_3) = 3\sigma^2, \ldots, V(x_T) = T\sigma^2$

$COV(x_1, x_2) = \sigma^2$; $COV(x_1, x_3) = \sigma^2$, $COV(x_1, x_T) = \sigma^2$

$COV(x_2, x_3) = 2\sigma^2$; $COV(x_2, x_4) = 2\sigma^2$, $COV(x_2, x_T) = 2\sigma^2$

$COV(x_{i-1}, x_i) = (i-1)\sigma^2$; $COV(x_{i-1}, x_{i+1}) = (i-1)\sigma^2$, $COV(x_{i-1}, x_T) = (i-1)\sigma^2$

$$VarCov(x^{H_0}) = \Sigma_1 = S_1{}'S_1 = \sigma^2 \begin{pmatrix} 1\ 1\ 1\ 1 \ldots 1 \\ 1\ 2\ 2\ 2 \ldots 2 \\ 1\ 2\ 3\ 3 \ldots 3 \\ .. \\ 1\ 2\ 3\ 4 \ldots t \end{pmatrix} where, S_1 = \sigma \begin{pmatrix} 1\ 1\ 1\ 1 \ldots 1 \\ 0\ 1\ 1\ 1 \ldots 1 \\ 0\ 0\ 1\ 1 \ldots 1 \\ .. \\ 0\ 0\ 0\ 0 \ldots 1 \end{pmatrix} \quad \ldots(15)$$

$\Sigma_1 = \sigma^2(\min(i,j))$.

Probability density of $(x_1, x_2, \ldots, x_T)^{H_0}$ is multi-variate normal MVN $(0, \Sigma_1)$

$$P(x_1, x_2, \ldots, x_T)^{H_0} = \left(\frac{1}{(2\pi)^{t/2} * (\Sigma_1)^{1/2}} e^{-\frac{(x)'\Sigma_1^{-1}(x)}{2}}\right) \quad \ldots(16)$$

III.2. PDF of a Stationary Process ($0 < \beta_t$<1 in equation (13))

$x_1 = e_1$;

$x_2 = \beta_2 * x_1 + e_2 = \beta_2 * e_1 + e_2$;

$x_3 = \beta_3 * x_2 + e_3 = \beta_3 * (\beta_2 * e_1 + e_2) + e_3 = \beta_2\beta_3 e_1 + \beta_3 e_2 + e_3$;

$x_4 = \beta_4 * x_3 + e_4 = \beta_4 * (\beta_2\beta_3 e_1 + \beta_3 e_2 + e_3) + e_4 = \beta_2\beta_3\beta_4 e_1 + \beta_3\beta_4 e_2 + \beta_4 e_3 + e_4$;

$x_T = \beta_T * x_{T-1} + e_t = .. = \beta_2\beta_3 \ldots \beta_T e_1 + \beta_3\beta_4 \ldots \beta_T e_2 + \beta_4\beta_5 \ldots \beta_T e_3 + .. + \beta_T e_{T-1} + e_T$;

$V(x_1) = \sigma^2$; $V(x_2) = \sigma^2[(\beta_2)^2 + 1]$, $V(x_3) = \sigma^2[(\beta_2\beta_3)^2 + (\beta_3)^2 + 1]$

$V(x_T) = \sigma^2[(\beta_2\beta_3..\beta_T)^2 + (\beta_3\beta_4 \ldots \beta_T)^2 + \ldots\ldots\ldots\ldots + (\beta_T)^2 + 1]$

$COV(x_1, x_2) = \sigma^2\beta_2$; $COV(x_1, x_3) = \sigma^2\beta_2\beta_3$, $COV(x_1, x_T) = \sigma^2\beta_2\beta_3 \ldots \beta_T$

$$COV(x_2, x_3) = \sigma^2[(\beta_2)^2\beta_3 + \beta_3]$$

$$COV(x_2, x_4) = \sigma^2[(\beta_2)^2\beta_3\beta_4 + \beta_3\beta_4]$$

$$COV(x_2, x_T) = \sigma^2[(\beta_2)^2\beta_3..\beta_T + \beta_3\beta_4..\beta_T]$$

$$COV(x_{i-1}, x_i) = \sigma^2[(\beta_2)^2\beta_3..\beta_i + \beta_3\beta_4..\beta_i]$$

$$COV(x_{T-1}, x_T) = \sigma^2[\beta_T(\beta_2\beta_3..\beta_{T-1})^2 + \beta_T(\beta_3\beta_4 \dots \beta_{T-1})^2 +..+ \beta_T(\beta_{T-1})^2 + \beta_T]$$

$$VarCov(x^{H_1}) = \Sigma_2 = S_2{}'S_2 \; where, S_2 = \sigma \begin{pmatrix} 1 & \beta_2 & \beta_2\beta_3 & \beta_2\beta_3\beta_4 & \dots & \beta_2\beta_3\beta_{4..}\beta_T \\ 0 & 1 & \beta_3 & \beta_3\beta_4 & \dots & \beta_3\beta_4\beta_{5..}\beta_T \\ 0 & 0 & 1 & \beta_4 & \dots & \beta_4\beta_5\beta_{6..}\beta_T \\ 0 & 0 & 0 & 1.. & \dots. & \beta_5\beta_6\beta_{7\dots}\beta_T \\ & & & & .. & \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix} \quad \dots(17)$$

If instead of time-varying stationary process, we assume the usual stationary process with $\beta_t = \beta$ for all t then the variance covariance matrix becomes:

$$VarCov(x^{H_1}) = \Sigma_2 = S_2{}'S_2; \; where, S_2 = \sigma \begin{pmatrix} 1 & \beta & \beta^2 & \beta^3 & \dots & \beta^{T-1} \\ 0 & 1 & \beta & \beta^2 & \dots & \beta^{T-2} \\ 0 & 0 & 1 & \beta & \dots & \beta^{T-3} \\ 0 & 0 & 0 & 1 & \dots. & \beta^{T-4} \\ & & & & .. & \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix} \quad \dots(18)$$

Probability density of $(x_1, x_2, \dots, x_T)^{H_1}$ is MVN $(0, \Sigma_2)$

$$P(x_1, x_2, \dots, x_T)^{H_1} = \left(\frac{1}{(2\pi)^{t/2} * (|\Sigma_2|)^{1/2}} e^{-\frac{(x)'\Sigma_2^{-1}(x)}{2}} \right)$$

III.3. The Test Criterion

If x: $(x_1, x_2, \dots, x_T)$ is a random sample of size 'T' generated out of a random walk process (H_0) from a PDF f(0,Σ_1), then, for some k>1,

$\frac{f(x, \Sigma_1)^{H_0}}{f(x, \Sigma_2)^{H_0}} \geq k$; where, f(0,$\Sigma_2$) is the PDF of a stationary process as defined in (13)

$$=> \frac{P(x_1, x_2, \dots, x_T / \beta_t = 1, \forall t)^{H_0}}{P(x_1, x_2, \dots, x_T / 0 < \beta_t < 1, \forall t)^{H_0}} \geq k \quad \dots(19)$$

$$\Rightarrow \frac{\left(\frac{1}{(2\pi)^{t/2} *(|\Sigma_1|)^{1/2}} e^{-\frac{(x)'\Sigma_1^{-1}(x)}{2}}\right)}{\left(\frac{1}{(2\pi)^{t/2} *(|\Sigma_2|)^{1/2}} e^{-\frac{(x)'\Sigma_2^{-1}(x)}{2}}\right)} \geq k$$

$$\Rightarrow \frac{(|\Sigma_2|)^{1/2}\left(e^{\frac{(x)'\Sigma_2^{-1}(x)}{2}}\right)}{(|\Sigma_1|)^{1/2}\left(e^{\frac{(x)'\Sigma_1^{-1}(x)}{2}}\right)} \geq k$$

$$\Rightarrow [\log(|\Sigma_2|) - \log(|\Sigma_1|)] + (x)'\Sigma_2^{-1}(x) - (x)'\Sigma_1^{-1}(x) \geq 2\log(k)$$

$$\Rightarrow [0 - 0] + (x)'\Sigma_2^{-1}(x) - (x)'\Sigma_1^{-1}(x) \geq 2\log(k)$$

$$\Rightarrow (x)'\Sigma_2^{-1}(x) - (x)'\Sigma_1^{-1}(x) \geq 2\log(k)$$

$$\Rightarrow (x)'\Sigma_2^{-1}(x) - (x)'\Sigma_1^{-1}(x) \geq k^* \qquad ...(20)$$

The test statistic (20) proposed in this paper is a linear combination of the non-central *chi*-squared variables. Its theoretical PDF is complex, and its derivation is beyond the scope of this paper. Therefore, it has been left for future research. Instead of theoretical PDF of the test statistic, the paper uses MCS to derive an empirical probability distribution of the test statistic and the threshold value or critical region is estimated from this empirical PDF.

Equation (20) suggests that when the data series x is indeed generated using random walk process, then the observed data x would look more probable (or higher likelihood) while fitting it using the gaussian normal distribution with variance-covariance matrix Σ_1 as defined in equation (15) than while trying to force-fit it with a variance-covariance matrix Σ_2 as defined in equations (17) or (18).

The proposed test statistic is $R_T = (x)'\Sigma_2^{-1}(x) - (x)'\Sigma_1^{-1}(x)$; where $x = (x_1, x_2, \dots, x_T)$.

Test criterion: Reject H_o if $R_T \leq k_T^{\alpha}$; where, k_T^{α} is the critical value; α is the size of type-I error such that P($R_T \leq k_T^{\alpha}$, when H_0 is true) =α; values of k_T^{α} are determined based on the MCS method.

III.4. Estimating the Threshold or Critical Value of the Test Statistic

The test statistic $R_T = (x)'\Sigma_2^{-1}(x) - (x)'\Sigma_1^{-1}(x)$ depends on β_ts in a complex way through Σ_2^{-1}, and can be shown as a linear combination of a set of variables which follows non-central *chi*-square distribution. Since the theoretical PDF is complex, its derivation is left for future research.

For example, if the sample size is set at 20, and for an instance of a set of β_t drawn from the uniform distribution (0,1), the test statistic after arithmetic adjustment becomes:

R_{20}=(0.5281x_1x_2 +1.9293 x_2x_3 +0.3783 x_3x_4 +0.4190 x_4x_5 +1.9011 x_5x_6 +1.8093 x_6x_7 +0.4858 x_7x_8 +0.5327 x_8x_9 + 0.1445x_9x_{10} +1.9514 $x_{10}x_{11}$ +0.5941x_{11}x1_2 +1.7462 $x_{12}x_{13}$ +0.0773 $x_{13}x_{14}$ +0.3003 $x_{14}x_{15}$ +1.3067 $x_{15}x_{16}$+0.2505 $x_{16}x_{17}$+0.2792 $x_{17}x_{18}$+1.2511 $x_{18}x_{19}$+1.1673 $x_{19}x_{20}$) - (0.46x_1^2 +1.00x_2^2+0.34x_3^2+0.38x_4^2+1.00x_5^2+0.99x_6^2+0.43x_7^2+0.46x_8^2+0.14x_9^2+1.00 x_{10}^2+0.51 x_{11}^2+0.98 x_{12}^2+0.08 x_{13}^2+0.28 x_{14}^2+0.88x_{15}^2 +0.23x_{16}^2 +0.26x_{17}^2 +0.86x_{18}^2 +0.83x_{19}^2) ...(21)

We know $X_iX_j= \frac{1}{4}(X_i+ X_j)^2$ - $\frac{1}{4}(X_i- X_j)^2$

And as X_i follows $N(0,\ \sigma^2)$, $(X_i+ X_j)^2 \sim \chi_1^2$; $(X_i- X_j)^2 \sim \chi_1^2$ also, $X_i^2 \sim \chi_1^2$

Therefore, the test statistic R_{20} in (21) can be written as a linear combination of a set of variables which are distributed as chi-squared (χ_1^2), and not necessarily independent. The derivation of the exact probability density function of the test statistic is very complex, and is not attempted here. Instead, the MCS-based empirical PDF is used to derive an empirical PDF of the test statistic. We start with a large set (N_1) of known non-stationary series of size 'T' generated using equation (14) and calculate the test statistic (R_T) for each of these N_1 non-stationary data series and calculate the empirical probability distribution of the test statistic for various quantiles.

While calculating the critical values of the test statistic, a known non-stationary dataset is contrasted with a stationary series using equation (19) with unknown parameters (β_t). However, β_t cannot be consistently estimated based on the sample observations ($x_2 = \beta_2 * x_1 + e_2;\ \ x_3 = \beta_3 * x_2 + e_3;$ ...). Therefore, to calculate critical values, β_ts are assumed here to be a random sample from a uniform distribution U(0.01,0.99).

III.5. Power of the Test Statistic

To estimate the empirical power of the new test criterion, we generate N_2 stationary series using equation (13) and calculate the power as follows:

Power of the test
= Probability (rejecting H_0 given H_1 is true) = P ($R_T \leq k_T^{\alpha}$ / when H_1 is true)
= (# of series identified by the test as stationary) / N_2
= proportionate of correctly identified stationary series.

Since we are estimating empirical power of the test based on a known set of stationary series, we need not be concerned about the specifics of alternative hypothesis. We can use the composite alternative hypothesis ($\beta_t < 1$ for all t); and determine the probability density of the test statistic under the alternative hypothesis, which may be complicated and is not required in this context.

Section IV
Empirical Analysis and Comparisons with a Few Other Unit Root Tests

IV.1. Empirical Probability Distribution of the Test Statistic

Here, 10 million nonstationary series for each of the 17 different sample sizes T (= 20, 25, 30, ...,100) are simulated by drawing random samples from a standard normal distribution with mean 0 and unit variance and then these observations are successively added using equation (14) (*i.e.,* $x_t = x_{t-1} + e_t$). The values of β_t, which are required to calculate Σ_2^{-1}, are assumed to be a random sample from a uniform distribution U(0.01,0.99). The test statistic $R = (x)'\Sigma_2^{-1}(x) - (x)'\Sigma_1^{-1}(x)$, is calculated for these 10 million simulated non-stationary individual data series and for each of the 17 category of sample sizes. The empirical probability distribution of the test statistic is given in Table 1.

Table 1: Empirical Probability Distribution of R_t for Non-Stationary Series

$k^{\alpha}_{T:}$	Random walk: Empirical pdf of R-test statistic: $P[R_T<k^{\alpha}_{T}]^{H0} = \alpha$										
	α: 1%	5%	10%	15%	20%	25%	30%	35%	40%	45%	50%
T:20	-4.81	-0.38	3.06	6.32	9.72	13.39	17.44	21.96	27.08	32.95	39.74
25	-4.16	1.76	6.76	11.51	16.40	21.65	27.41	33.84	41.09	49.36	58.93
30	-2.64	5.75	13.25	20.43	27.86	35.86	44.65	54.47	65.54	78.17	92.75
35	-0.85	10.26	20.18	29.61	39.38	49.86	61.34	74.16	88.68	105.23	124.37
40	1.87	16.47	29.73	42.46	55.64	69.84	85.47	102.90	122.58	145.14	171.24
45	3.59	20.08	34.96	49.20	63.95	79.86	97.37	116.86	138.99	164.30	193.59
50	6.74	26.74	44.54	61.39	78.72	97.29	117.60	140.17	165.70	194.87	228.69
55	8.55	30.41	50.09	68.66	87.75	108.13	130.39	155.19	183.08	214.91	251.65
60	13.33	39.64	63.28	85.65	108.55	132.99	159.63	189.18	222.40	260.14	303.62
65	19.13	51.90	81.66	109.89	138.95	169.94	203.81	241.49	283.73	331.68	386.71
70	23.65	59.98	92.92	124.37	156.77	191.35	229.14	271.08	318.45	372.07	433.59
75	29.51	71.11	108.87	144.77	181.76	221.33	264.65	312.71	366.74	428.31	499.08
80	34.39	80.22	121.79	161.20	201.99	245.83	293.61	346.67	406.33	474.30	552.35
85	42.81	96.25	144.67	190.80	238.34	289.39	345.16	407.31	477.40	557.01	648.68
90	50.51	110.45	165.06	217.11	271.05	328.76	391.93	462.30	541.56	632.05	736.10
95	56.67	121.51	180.58	236.69	294.82	357.13	425.42	501.44	587.03	684.94	797.85
100	67.11	140.39	206.89	270.47	336.23	406.66	483.86	569.94	667.00	777.70	905.89

$k^{\alpha}_{T:}$	Random walk: Empirical pdf of R-test statistic: $P[R_T<k^{\alpha}_{T}]^{H0} = \alpha$									
	55%	60%	65%	70%	75%	80%	85%	90%	95%	99%
T:20	47.71	57.14	68.50	82.34	99.51	121.66	151.66	195.90	275.13	470.53
25	70.14	83.37	99.24	118.57	142.68	173.71	215.55	277.43	388.53	662.01
30	109.79	129.94	154.06	183.52	220.07	267.04	330.79	424.59	593.24	1008.64
35	146.67	173.03	204.57	243.03	290.93	352.54	435.96	559.28	779.49	1321.08
40	201.75	237.89	281.10	333.85	399.59	484.04	598.09	766.50	1069.18	1812.99
45	227.84	268.31	316.91	376.17	449.99	545.08	673.45	862.20	1201.66	2035.56
50	268.23	314.86	370.80	439.11	523.94	633.23	780.74	998.95	1390.40	2354.68
55	294.69	345.48	406.27	480.44	572.88	691.50	852.28	1088.75	1514.64	2563.20
60	354.42	414.31	486.10	573.49	682.44	822.22	1010.97	1288.91	1790.70	3023.64
65	450.82	526.25	616.63	726.67	863.56	1039.70	1277.58	1629.10	2261.84	3815.63
70	505.18	589.78	690.75	813.48	966.17	1162.64	1428.12	1819.55	2525.39	4263.04
75	581.31	678.44	794.38	935.41	1111.23	1337.23	1642.16	2092.08	2901.89	4893.33
80	643.28	750.52	878.74	1034.75	1229.35	1479.76	1817.35	2316.41	3214.85	5421.44
85	755.22	881.48	1032.07	1215.87	1444.52	1738.51	2136.37	2723.13	3778.03	6371.56
90	857.75	1001.20	1173.26	1382.39	1643.06	1978.76	2431.67	3100.53	4300.54	7266.24
95	929.71	1085.45	1271.49	1498.52	1781.28	2144.44	2636.17	3361.16	4665.63	7875.73
100	1055.38	1231.76	1443.68	1701.75	2022.99	2435.75	2994.55	3816.87	5294.79	8943.34

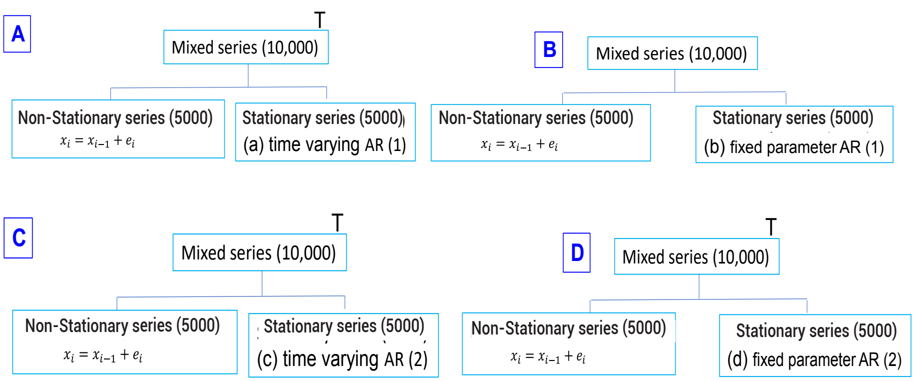
Note: The table above presents critical values k^{α}_{T} of the empirical distribution of proposed test statistic (R_T) corresponding to specified significance levels (α). The computation has been carried out for different sample sizes in the range of 20 to 100.

Source: Author's calculations.

IV.2. Comparison of the Power of Unit-Root Tests

The performance of the new unit-root test, denoted as R-test for notational convenience, is empirically compared with other commonly used univariate unit-root tests *viz*., (1) Dickey and Fuller's ADF test (Dickey and Fuller, 1979), (2) Phillips-Perron test (Phillips and Perron, 1988), (3) KPSS test (Kwiatkowski, Phillips, Schmidt and Shin, 1992), (4) Elliot, Rothenberg, and Stock (ERS) test, (5) Zivot and Andrews (ZA) test, (6) Schmidt and Phillips (SP) test, (7) Pantula, Gonzales-Farias and Fuller (PGFF) test, and (8) Breitung's variance ratio (BVR) test.

Each of the selected unit root test is applied on the time series data of various sample sizes consisting of stationary series as well as non-stationary series. We observe as to how many of these series are correctly identified as stationary or non-stationary series. The stationary series are generated in four different ways using (a) AR(1) models with time-varying coefficients (α_t, γ_t), (b) AR(1) models with fixed coefficients (β) where 0< β<1, (c) AR(2) models with time-varying coefficients (α_t, γ_t), and (d) AR(2) models with fixed coefficients (β) where 0< β<1.



IV.2.1. Sample Generation

To compare the performance of the unit-root test, we use 16 different sample sizes (T=25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, 80, 85, 90, 95, and 100). For each of the selected sample size (T), 10,000 random samples were generated consisting of 5,000 stationary and 5,000 non-stationary series.

The stationary series of various sample sizes *viz*., T(=25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, 80, 85, 90, 95, and 100) were constructed as follows:

(a) AR(1): time-varying coefficients: using model $x_t = \beta_t * x_{t-1} + e_t$; where $0 < \beta_t < 1$ for all t=1(1)T and assuming $x_0 = 0, x_1 = e_1$, and $e_t \sim N(0,1)$;

(b) AR(1): fixed coefficient: using a AR(1) model *i.e.*, $x_t = \beta\, x_{t-1} + \varepsilon_t$ with fixed parameter (β), $x_0 = 0$, $e_t \sim N(0,1)$; then for different positive parameter values ($fixed\,\beta$ = 0.10, 0.20,…0.90,0.95,0.99);

(c) AR(2): time-varying coefficients $x_t = \alpha_t * x_{t-1} + \gamma_t * x_{t-2} + e_t$; where $0 < \alpha_t < 1$ and $0 < \gamma_t < 1$ and $\alpha_t + \gamma_t < 1$; for all t=1(1)T and assuming $x_0 = 0, x_1 = e_1$, and $e_t \sim N(0,1)$;

(d) AR(2): fixed coefficient: using a AR(2) model *i.e.*, $x_t = \beta\, x_{t-1} + \gamma\, * x_{t-2} + \varepsilon_t$ with fixed parameters (β and) within a derived series; then for different positive values of the fixed parameter (β and γ = (0.10, 0.20,…0.90,0.95 0.99 and β + γ <1.

The non-stationary series was constructed by drawing random samples from a standard normal distribution with mean 0 and unit variance and then successively adding the observations using (14).

IV.2.2. Empirical Results – Time-Varying Parameters: AR (1) and AR (2)

At 5 per cent significance level (α), the power of the unit-root tests for various size of the sample (T) and for stationary series generated using (a) AR (1) with different parameter β; (b) time-varying AR (1) model; (c) AR (2) model; and (d) time-varying AR (2) are calculated, and shown in Charts 1 and 2. Despite setting the significance level at 5 per cent, some of the tests produced higher Type I errors in the simulation exercise. Therefore, the empirical [(1-Type I error) + (1-Type II error)]/2 or proportion of correctly identified series by the tests is also presented in Annex to corroborate the effectiveness of the tests. Only for the KPSS test, the null hypothesis is that the series is stationary, while for all other tests the null hypothesis is that the series is non-stationary.

It is observed that the power of the proposed test exhibits superior performance, but it varies with the size of sample and for different β values.

Chart 1 (a to f): Power of Tests: Comparisons of Unit Root Test: AR (1)

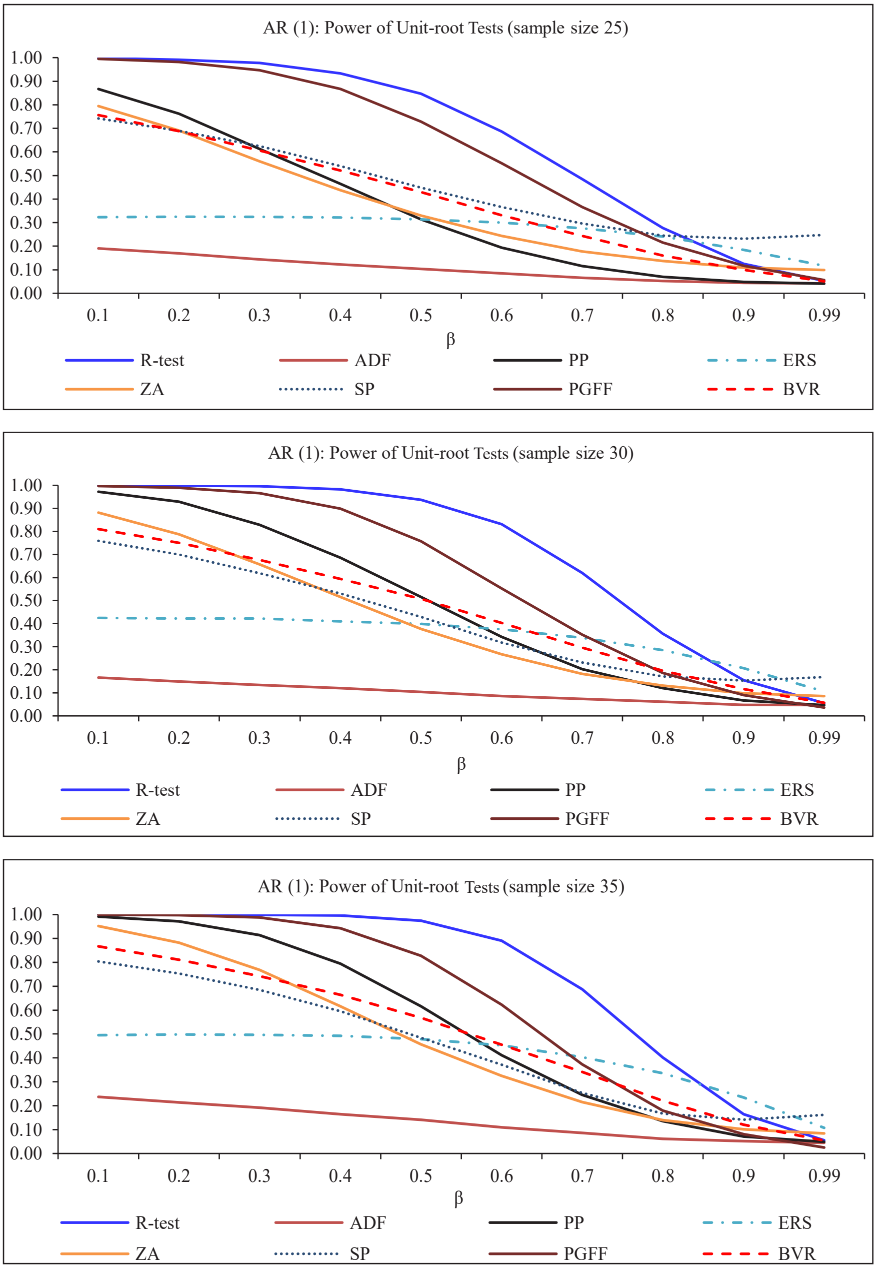
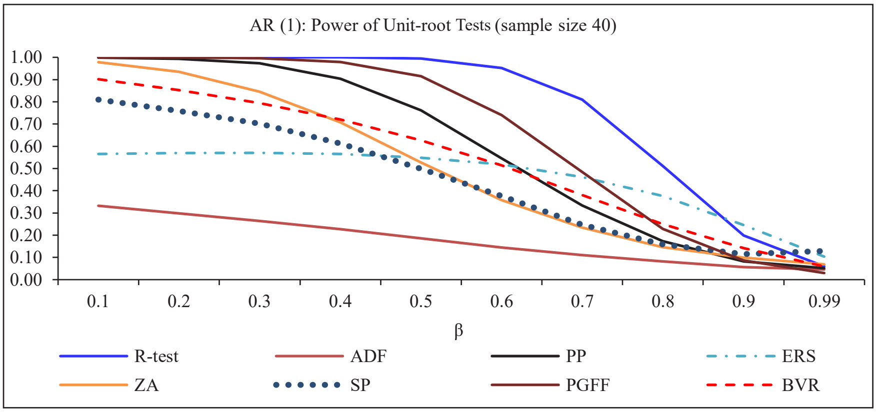
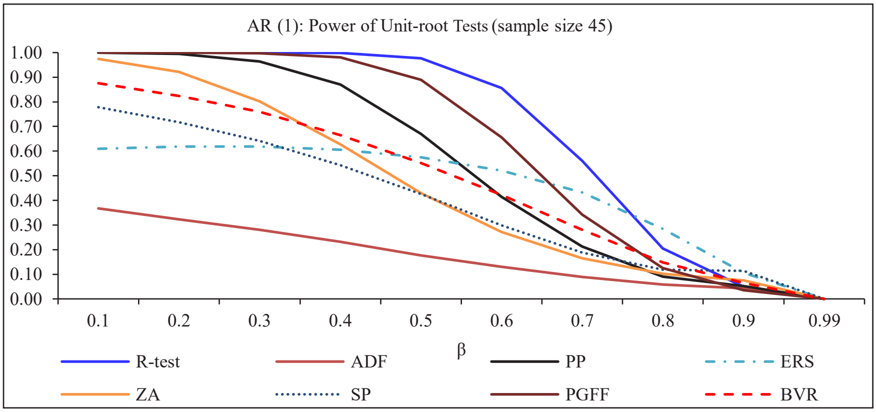
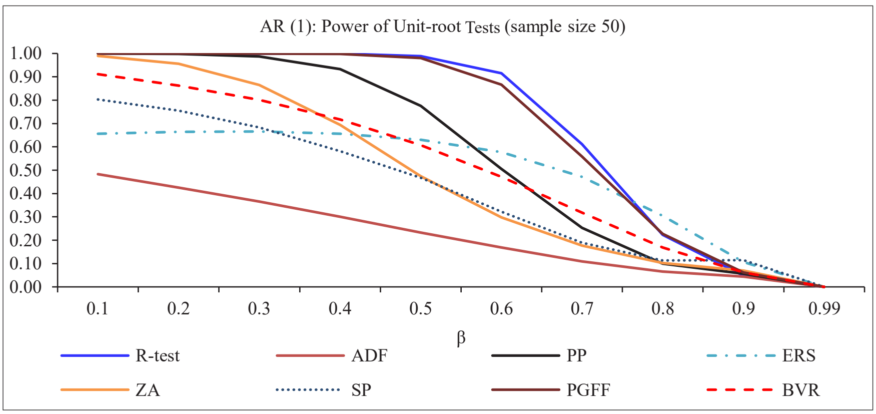


Chart 1 (a to f): Power of Tests: Comparisons of Unit Root Test: AR (1) *(Concld.)*

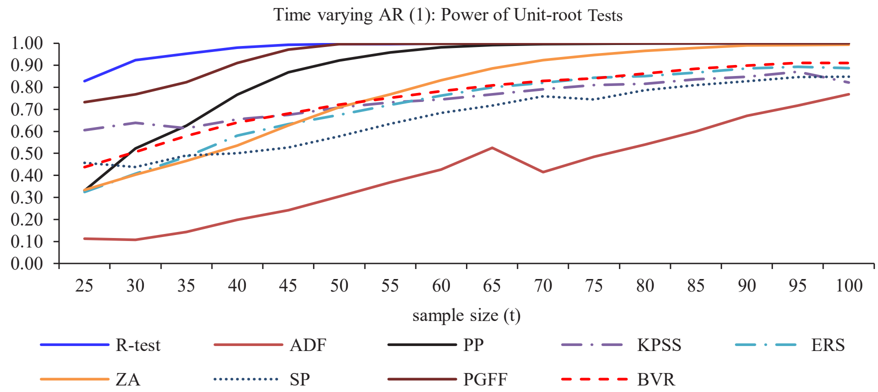






Source: Author's calculations.

Chart 2: Power of Tests: Comparisons of Unit Root Test: Time-Varying AR (1)



Source: Author's calculations.

It is observed that the power or the ability to correctly identify a stationary series is significantly higher for the new test than for the other selected unit-root tests when the sample size is under 50. For large sample sizes, the new unit-root test mirrors either improved or on-par efficiency as compared to the other tests.

Section V
Conclusions

A new test criterion is developed in this paper to test the presence of unit root in a zero-mean time series with no deterministic trend and no structural break. The test statistic has been developed with the assumption that if a given data series is generated out of a random walk process, then it will result in a better fit when the PDF of a random walk process is applied to it, rather than force-fitting it with the PDF of a stationary process.

The proposed unit-root test is generic in nature and is effective to any time series irrespective of the practitioner's assumption on the time-variant coefficients or orders of AR/MA processes.

To estimate the empirical PDF and critical values of the test statistic, the MCS method is used, wherein a large set (10 million) of known non-stationary series of various sample sizes (ranging from 20 to 100) are generated. The test statistic is then calculated for the generated data series, and is used as a reference. The performance of the proposed new unit-root test statistic is empirically compared with other commonly used univariate unit-root tests *viz.*, ADF test, PP test, KPSS test ERS test, Zivot and Andrews test, Schmidt and Phillips (SP) test, PGFF test, and BVR test. For this purpose, all these selected unit root tests are applied individually to a large set of stationary as well as non-stationary data of varied length, which are synthetically generated using time-varying as well as fixed AR(1) and AR(2) models for stationary data, and random walk model for non-stationary data.

It is observed that for small samples, the power of the proposed test is significantly higher than the other selected unit-root tests, particularly when the sample size is under 50. For large samples, the effectiveness of the proposed test is still higher than most of the selected tests and is on-par with the remaining ones.

The higher power of the test is demonstrated only under the case of no trend and no structural breaks. However, most of the time series data have trends, which bring in another source of non-stationarity. The proposed test can be developed further to account for the trend and structural breaks. Further, although the new test demonstrates improved performance when error terms are correlated, the specific design and derivation of the PDF of the test statistic with correlated error terms is another area for future work.

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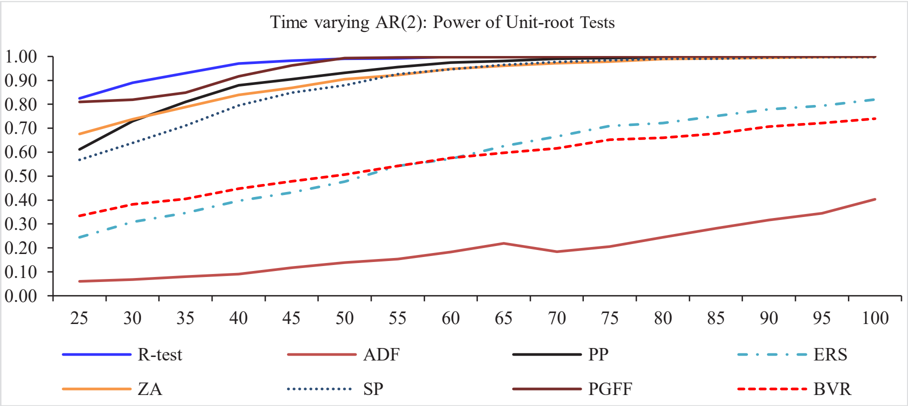
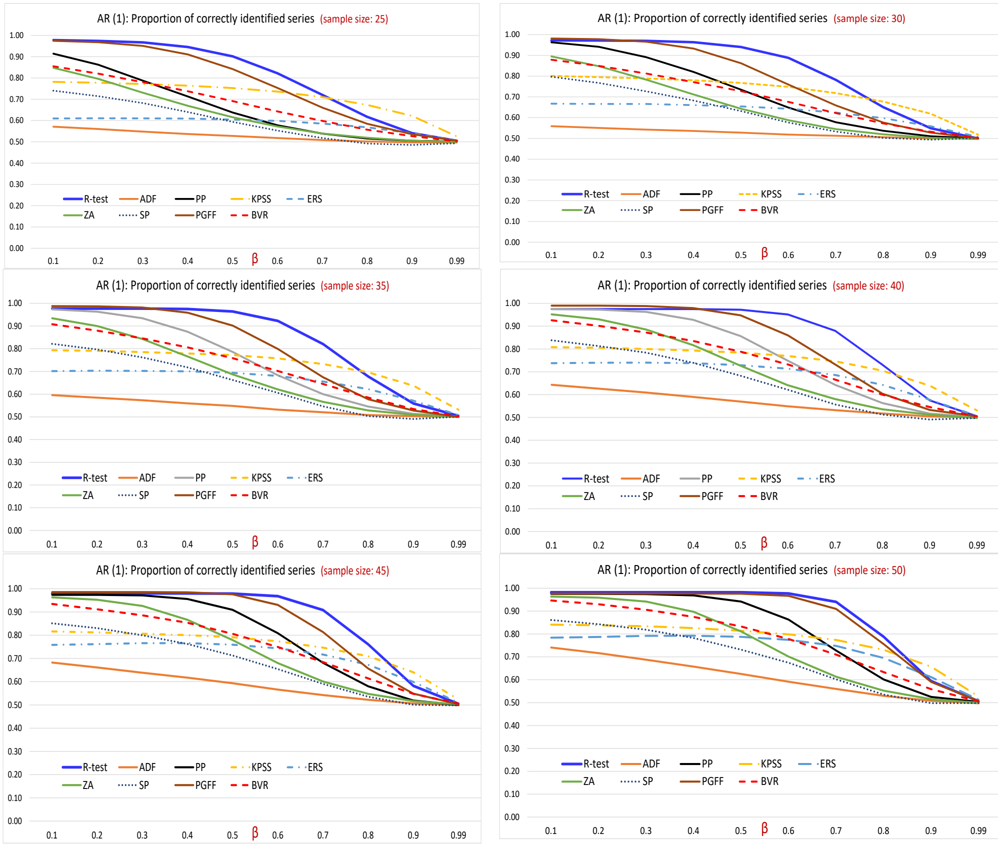
Annex

Chart A1: Power of Tests: AR (2) - Time-Varying



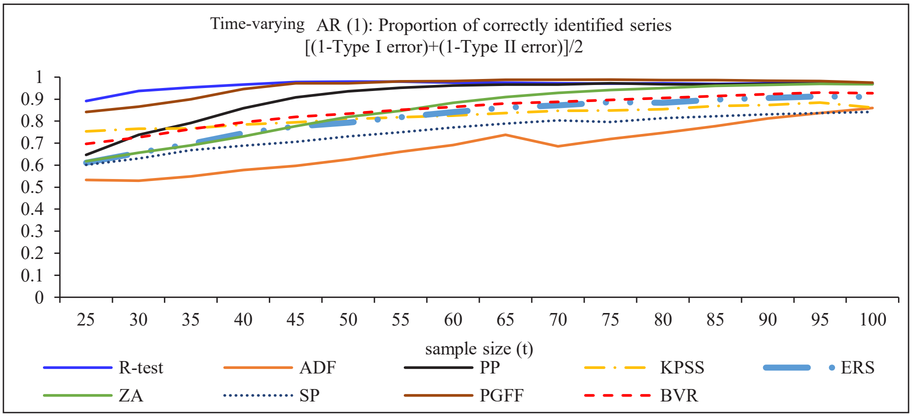
Source: Author's calculations.

Chart A2: Proportion of Correctly Identified Series: 5000 AR (1) + 5000 Random Walk Series



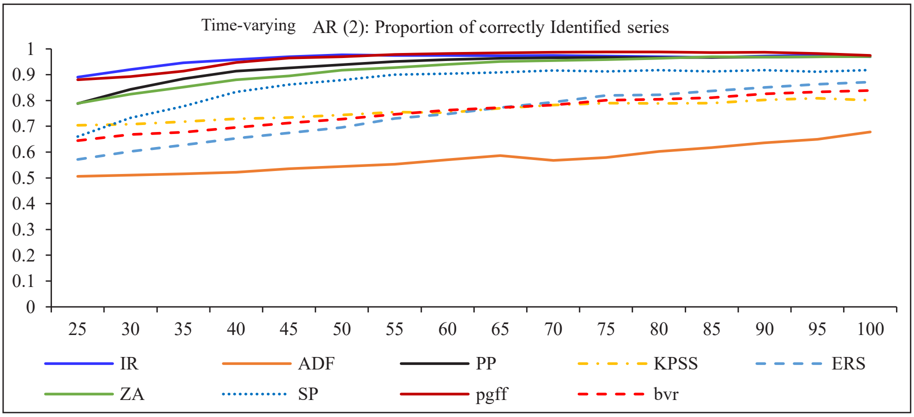
Source: Author's calculations.

Chart A3: Proportion of Correctly Identified Series: Time-Varying AR(1) and Random Walk



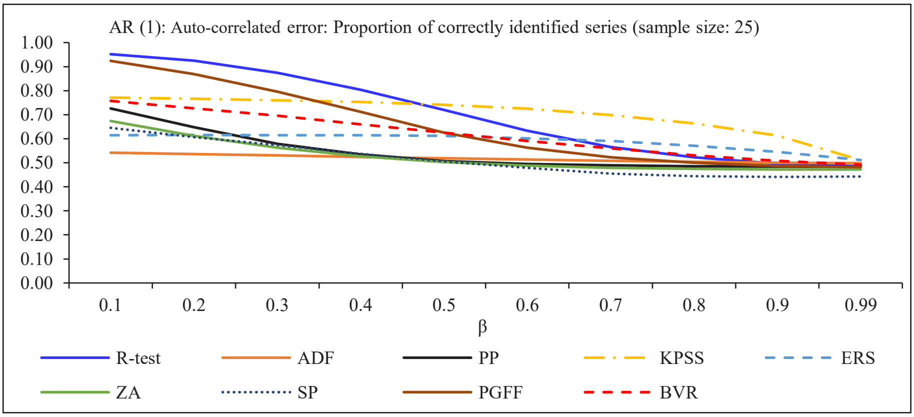
Source: Author's calculations.

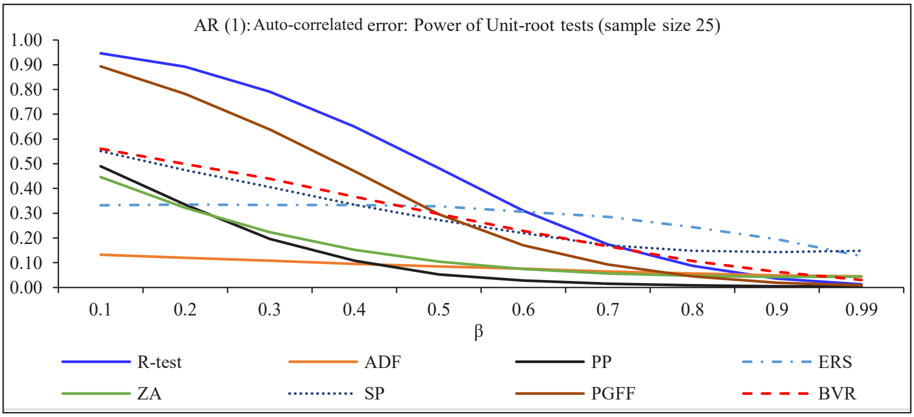
Chart A4: Proportion of Correctly Identified Series: AR (2) - Time-Varying and Random Walk



Source: Author's calculations.

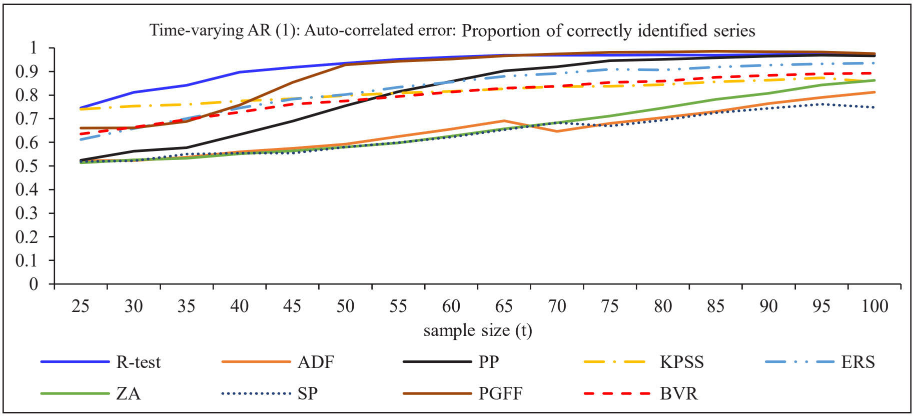
Chart A5: AR (1) - With Auto-Correlated Error

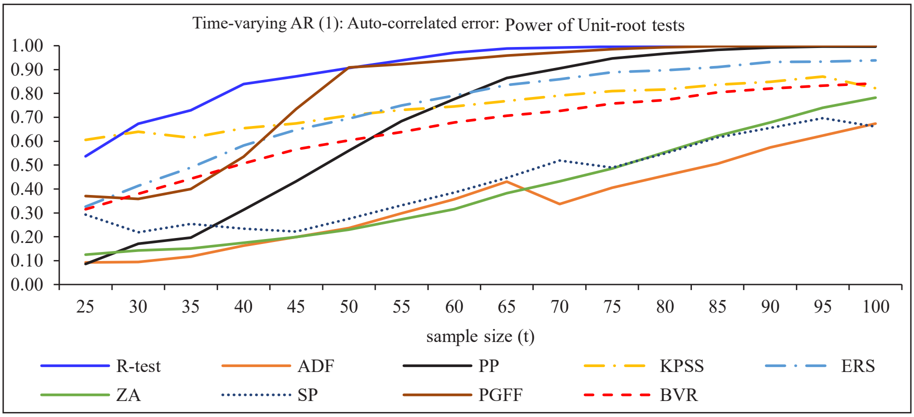




Source: Author's calculations.

Chart A6: Time-Varying AR (1) - With Auto-Correlated Error





Source: Author's calculations.